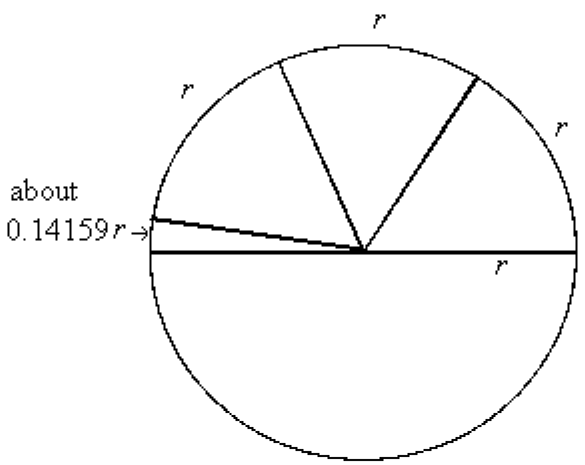
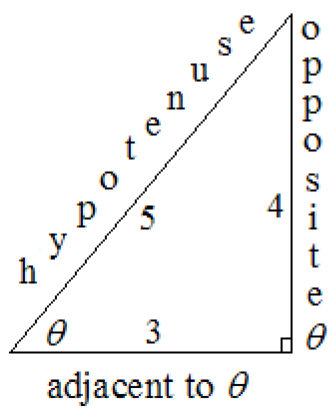
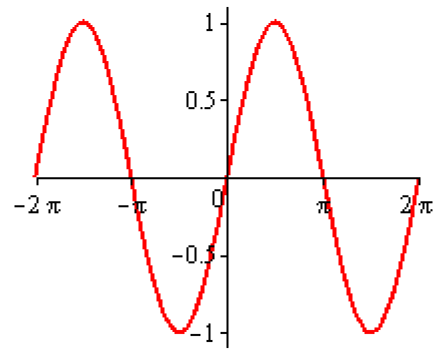
Mathematics Post- Secondary Preparation Package 2020 (MP^3)



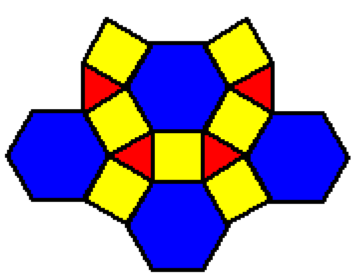


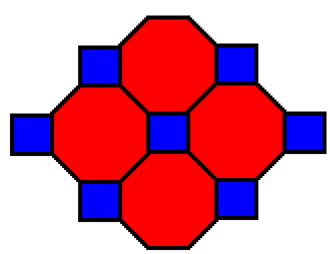


(i) $\cot^2(\theta)+1=\boxed{}$ (ii) $\tan(-\theta)=\boxed{}$ (iii) $\tan\left(\frac{\pi}{2}-\theta\right)=\boxed{}$

(iv) $\tan(A\pm B)=\boxed{}$ (v) $\tan(2A)=\boxed{}$

$$ax^2+bx+c=0 \Leftrightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$





To students starting first year college or university in the fall of 2020:

Here is a sad truth from a veteran math professor. One of the biggest fears of many students entering university or college is that they will struggle because their math skills are rusty. Even many students whose high school math marks were **high** come to class a little nervous.

Some math professors, or in fact, professors teaching ANY course that uses or builds on math, will assume you have your high school math down pat and at your fingertips. But you may not. You closed your books in June (or January!) and haven't solved a quadratic equation or found the hypotenuse of a right triangle or sin(30°) since **forever!** Your basic math skills and knowledge are surrounded by neural cobwebs.

Well, we have a tool for you: the Math Post-Secondary Preparedness Package = **MP^3!**

This package was originally developed in 2009 by Professor Hosh Pesotan and me (Jack Weiner) and two University of Guelph senior math students, Sylvia Nguyen and George Hutchinson.

Well, with the school closures in March and the scramble to transition to online learning, MP^3 is more relevant that ever. So, I asked Mathematics and Physics teacher (and my former student) Sarah Stubbs to come on board. She graciously agreed, and we have updated MP^3.

What is in the MP^3?

- A FREE resource to help you be fully prepared to succeed in your first year courses, whether math courses or a courses that rely heavily on math such as chemistry and physics.
- 11 problem sets (mostly 5 to 10 types of problems in each)
- Examples from grade 8 math to calculus
- Full solutions
- Notes and common errors to be wary of for each example
- Additional practice questions (with answers of course)
- Links for additional resources. Every question has a **clickable** link to a YouTube video and a relevant website.

More on Clickable Links: With rare exception, each problem solution is one page long. At the bottom are two links, one to a YouTube video on the topic of that problem and one to a website about that topic. Each video and website we chose had to be friendly, accurate, relevant of course, and even fun. To access these, place your cursor on the link. Then simultaneously hold down the "ctrl" key and click your mouse. A browser will open up and take you directly to the correct webpage. Sarah, Hosh, and I independently checked each link. All the links work. And the sites are safe.

Having said that, here is a message from Google*: "**Google's Safe** Browsing technology examines billions of URLs per day looking for unsafe websites. ... When we detect unsafe* *sites, we show warnings on **Google** Search and in web browsers. You can search to see whether a website is currently dangerous to visit."*

How to Use MP^3

Start working through the package four to eight weeks before you plan to return to school so that you hit the ground running, mathematically, right from the outset. When that first crucial week of classes arrives, you will NOT get stuck in class on new math or physics or chemistry or… because the prof assumed and went too fast through some fuzzy high school math.

- Spend 1 to 2 hours a day (okay, maybe 3 for some problem sets) for 11 days, one per problem set.
- Really try to solve each problem before checking the solutions. (Okay, take a peak if you must but after getting a starting hint, continue on your own.)
- Perhaps connect with a friend via Skype or Facetime or Zoom and help one another as you work through the package.
- Keep this as a review package throughout your program.

Why did we create this?

- We don't want you getting stuck on new concepts because you're fuzzy with high school math on which the new concepts build.
- With a solid foundation, you will approach the post-secondary math with confidence. A large part of success in math comes from the justified belief that "I can do this!"
- We care about you and we're passionate about math and your success in math! In life, for that matter.

We can't emphasize this enough. Please don't leave this till school starts. Once you begin first year, whether online or on campus, you will have between 18 and 30 hours of classes and labs, lots of challenging homework and assignments, tests almost immediately, plus new life skills to learn. You won't have time for the Preparedness Package. Work through this package in four to eight weeks before classes start.

The three of us have looked over all of the solutions, so we are confident that errors are rare. However, if you find one, please let us know. Email either me or Sarah and we will fix it. And give you credit. If you find a link that doesn't work, again, please let us know.

IMPORTANT: MP^3 is a free resource. We are making it available as a free download from several sources. You are welcome to share it with anyone you think will find it useful as well as your math teachers. If they like it, we would love them to share it with all their senior students. We want MP^3 helping far and wide.

Good luck and best wishes!

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A little bit about the authors:

Professor Jack Weiner and Professor Hosh Pesotan taught at the University of Guelph for a combined total of over 65 years! But to keep us young at heart and in mind, we enlisted the aid of …

…Sarah Stubbs. Sarah completed a Bachelor of Science at the University of Guelph with a focus on Mathematics, Physics, and Studio Arts. She then completed a Bachelor of Education at Queen's University specializing in high school Math and Visual Arts. Sarah is a Math and Physics tutor, professional painter, and founder of an education organization called Life Gears Academy that runs hands-on STEM (Science, Technology, Engineering, and Math) camps programs for youth. Sarah has a great passion for making a difference for others through education, mentorship, and art. Feel free to reach out to her for anything!

And little bit about Suzanne Tyson of HigherEdPoints suzanne@higheredpoints.com

In 2008, Suzanne and I (Jack) asked each other, "How can we help students succeed in math?" We decided a priority was to help students successfully transition from high school mathematics to post-secondary mathematics. And MP^3 came to life. Suzanne nurtured this project from beginning to end. She edited, contributed to the design, and supervised the creation of the print edition from start to finish. When I decided that an update for 2020, with all the current isolation challenges, was an even greater priority, Suzanne enthusiastically came on board. Again, she has nurtured the project from beginning to completion. Since this is a non-profit enterprise, she has been contributing her time, effort, and considerable expertise pro bono because she believes in the project. She ***wants*** to help students succeed. I am honoured to have worked with her and proud to have her as friend and colleague. I speak for myself and Hosh and Sarah: thank you, Suzanne!

Mathematics Post- Secondary Preparation Package 2020 Brief Table of Contents

Problem Sets

And then the Solutions!

Part 1: Brushing up on Numerical Skills

1) Adding and Subtracting Fractions

Problem: Evaluate **without a calculator!**

(i) $\frac{2}{3}+2\frac{5}{6}$ (ii) $3\frac{2}{5}-2\frac{3}{4}$ (iii) $\frac{1}{6a}+\frac{2}{3a}-\frac{5}{12a}$

2) Multiplying and Dividing Fractions

Problem: Evaluate **without a calculator!**

(i) $\frac{5}{3}\times\frac{12}{25}$ (ii) $1\frac{2}{5}\times\frac{3}{4}$ (iii) $3\times\frac{4}{5}$ (iv) $\frac{\left(\frac{5}{3}\right)}{\left(\frac{25}{12}\right)}$ (v) $\frac{\left(\frac{4}{3}\right)}{5}$ (vi) $\frac{4}{\left(\frac{3}{5}\right)}$

3) Working with Decimals.

Problem: Evaluate **without a calculator!**

(i) $1.02+.023$ (ii) $1.02-2.57$ (iii) $1.2\times.05$ (iv) $\frac{4.291}{3}$

(v) $\frac{4.291}{0.3}$ (vi) $\frac{0.004291}{0.03}$

4) Roots and Radicals

Problems: 1) Write as simplified mixed radicals:

(i) $\sqrt{40}$ (ii) $2\sqrt{27}$ (iii) $\sqrt{x^4y^7}$ (iv) $\sqrt[3]{x^4y^7}$

2) Write as entire radicals: (i) $3\sqrt{2}$ (ii) $\frac{4}{9}$ (iii) $xy^4\sqrt{xy}$ (iv) $xy^4\sqrt[3]{xy}$

3) Evaluate: (i) $\sqrt{121}$ (ii) $\left(\frac{27}{64}\right)^{2/3}$ (iii) $32^{1/5}$ (iv) $(-32)^{1/5}$ (v) $(-64)^{1/6}$

5) Absolute Value

Problems: 1) Evaluate: (i) $|10|$ (ii) $|-10|$ (iii) $|0|$

2) Write $|x|$ without absolute value bars if (i) $x > 0$ (ii) $x < 0$

3) Draw the graph of $y = |x|$

Part 2: Lines and Slopes

1) Finding the Slope of a Line

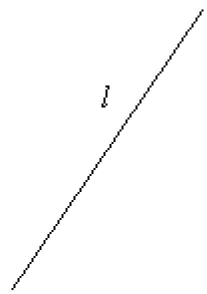
Problem: Find the slope of the line joining (2,5) to (7,4).

2) Parallel and Perpendicular Slopes

Problem: Find the slope of the line

(i) parallel (ii) perpendicular

to a line *l* with slope 2.

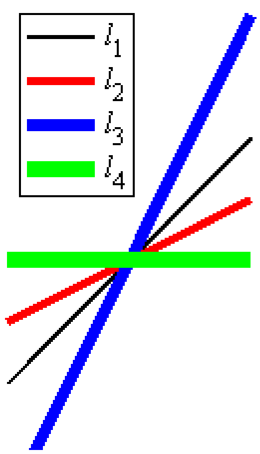


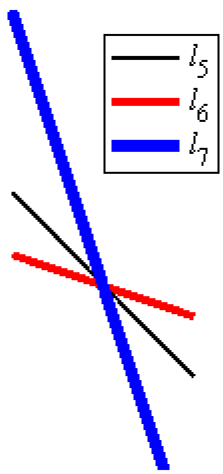
3) Interpreting Slope

Problem:

(i) Match the slopes $0, \frac{1}{2}, 1, 2$ with the lines l_1, l_2, l_3, l_4.

(ii) Match the slopes $-\frac{1}{3}, -1, -3$ with the lines l_5, l_6, l_7.





4) Finding the Slope and Intercepts From the Equation of a Line

Problem: Find the slope and *x* and *y* intercepts for each of the following lines:

(i) $y = -2x + 5$ (ii) $6x + 2y = 1$ (iii) $y = 5$ (iv) $x = 1$

5) Finding the Equation of a Line Given Two Points

Problem: Find the equation of the line joining (4,5) to (7,14). Draw the graph.

6) Finding the Equation of a Line Given the Slope and a Point

Problem: Find the equation of the line with slope $m = -2$ passing through the point $(-4,5)$. Draw the graph.

7) Graphing Linear Inequalities

Problem: Show by shading the region in the $x\,y$ plane that satisfies
(i) $x + y \leq 3$ (ii) $2x - y > 4$.

Part 3: Algebraic Skills

1) Adding and Subtracting Like Terms

Problem: Simplify: (i) $4st^2 + 2t^2s$

(ii) $(x^2 - 3xy + 7x - 1) + (2x^2 - xy - 3y - 4)$

(iii) $3(x+z) + 7(x+z) - y(x+z)$

2) Multiplying Binomials

Problem: Expand: (i) $(3x+1)(2x-5)$ (ii) $(2a+3b)^2$

3) Multiplying Binomials and Trinomials

Problem: Expand: (i) $(x^2+3x+1)(2x-5)$ (ii) $(a+b+c)^2$

4) Expanding $(a \pm b)^3$

Problem: Expand: (i) $(a+b)^3$ (ii) $(a-b)^3$

5) Factoring Easy Trinomials

Problem: Factor: (i) x^2+5x+4 (ii) x^2+3x-4 (iii) $6x^2+17x+5$ (iv) $6x^2-13x-5$

6) Factoring Less Easy Trinomials Using the Quadratic Formula

Problem: Factor: (i) x^2+3x+1 (ii) $6x^2-5x-2$

7) Factoring Difference of Squares: $a^2 - b^2 = (a-b)(a+b)$

Problem: 1) Factor: (i) x^2-9 (ii) $x^4-(y+1)^2$

2) Rationalize the denominator: $\dfrac{1}{\sqrt{x}-4}$

8) Factoring Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Problem: (i) Factor: $8x^3 - 27$

(ii) Rationalize the denominator: $\dfrac{1}{\sqrt[3]{x} - 2}$

9) Factoring Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Problem: (i) Factor: $8x^3 + 27$

(ii) Rationalize the denominator: $\dfrac{1}{\sqrt[3]{x} + 2}$

10) Factoring $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + ... + ab^{n-2} + b^{n-1})$

Problem: Factor: $x^5 - y^5$

11) Factoring $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3 + ... - ab^{n-2} + b^{n-1})$
and *n* MUST BE ODD!

Problem: Factor: $x^5 + y^5$

12) The Factor Theorem: Part 1

Problem: Factor the expression $x^3 - 4x^2 + x + 6$.

13) The Factor Theorem: Part 2

Problem: Find all the **rational** roots of $2x^3 - 5x^2 - 4x + 3$.

14) Polynomial Division

Problem: Divide $x^3 - 5x^2 + x + 6$ by $x - 4$; express your answer $\dfrac{x^3 - 5x^2 + x + 6}{x-4}$ both in the form $\dfrac{x^3 - 5x^2 + x + 6}{x-4} = \text{Quotient} + \dfrac{\text{Remainder}}{\text{Divisor}}$

and $x^3 - 5x^2 + x + 6 = \text{Quotient} \times \text{Divisor} + \text{Remainder}$.

15) Solving for the Roots of a Polynomial

Problem: Solve: (i) $x^2 - 5x + 6 = 0$ (ii) $x^2 - 5x + 1 = 0$ (iii) $x^3 - 7x = -6$

16) Solving "Factored" Inequalities: Numerators Only

Problem: Solve: (i) $(x+1)(x-1)(x-2) \geq 0$ (ii) $(x+1)^2 x(x-1)^3 > 0$

17) Solving "Factored" Rational Inequalities:

Problem: Solve: (i) $\dfrac{(x+1)(x-2)}{(x-1)} \geq 0$ (ii) $\dfrac{(x+1)^2(x-1)^3}{x} > 0$

18) Completing the Square

Problem: Complete the square: (i) $x^2 - 8x + 25$ (ii) $3x^2 + 36x - 17$ (iii) $-2x^2 + 3x + 1$

19) Adding and Subtracting Rational Expressions

Problem: By getting a common denominator, simplify the following expressions:
(i) $\dfrac{1}{3x} - \dfrac{1}{2y} + \dfrac{1}{6z}$ (ii) $\dfrac{2x+1}{x-1} - \dfrac{x+1}{x+2} - \dfrac{5x+4}{x^2+x-2}$

20) Multiplying and Dividing Rational Expressions

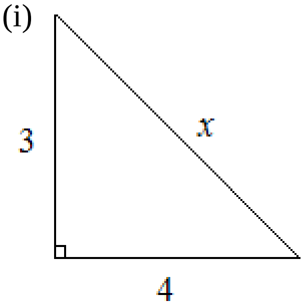
Problem: Simplify the following rational expressions:
(i) $\dfrac{(x^2-16)}{(x-4)^3} \times \dfrac{(x^2-4x)^2}{x^3+64}$ (ii) $\dfrac{x^2+5xy+4y^2}{x^2+4xy+4y^2} \div \dfrac{x^2+xy}{x^2+2xy}$

Part 4: Geometry

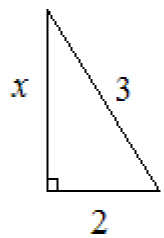
1) Pythagorean Theorem

Problem: In the following diagrams find the value of the unknowns:

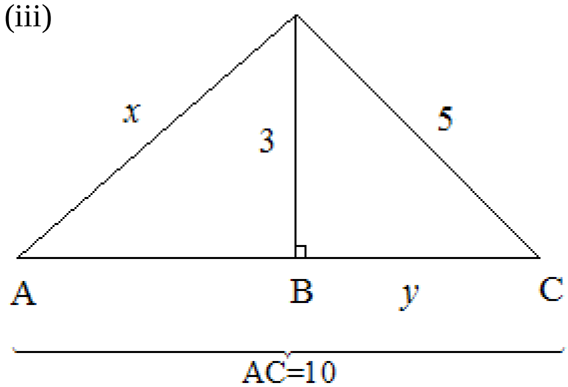
(i)



(ii)



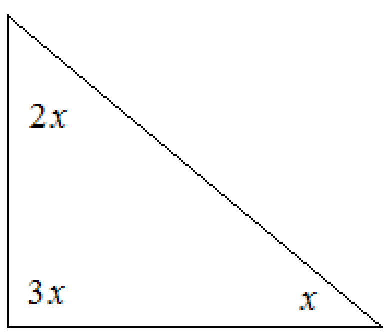
(iii)



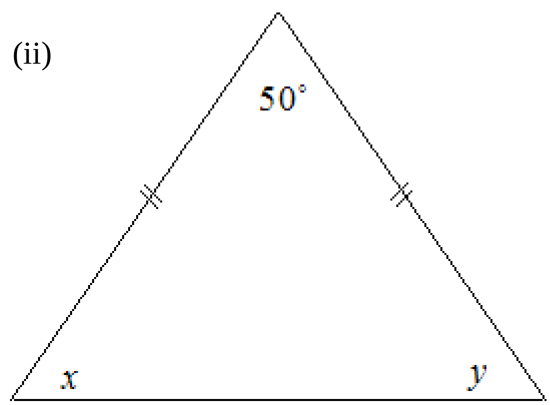
2) Angles in a Triangle

Problem: Find the values of angles *x* and *y* (in degree measure) from the following diagrams:

(i)



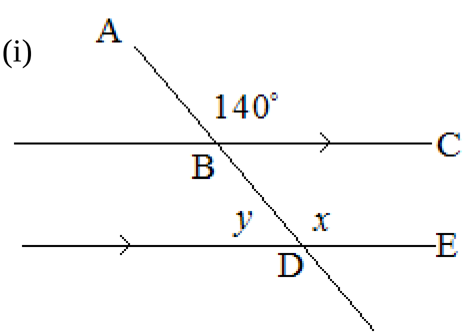
(ii)



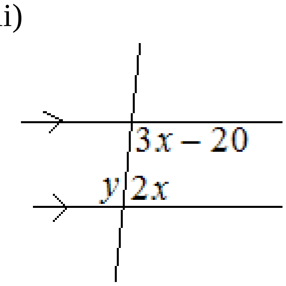
3) The Parallel Line Theorem

Problem: Find the values in degrees of *x* and *y* in the following diagrams:

(i)

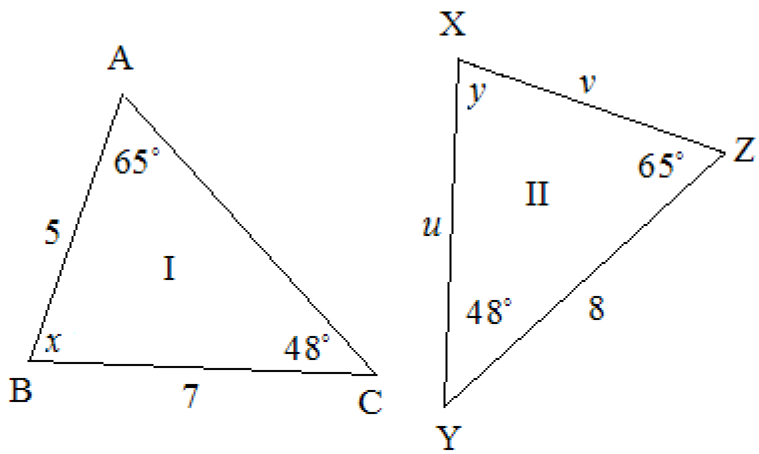


(ii)



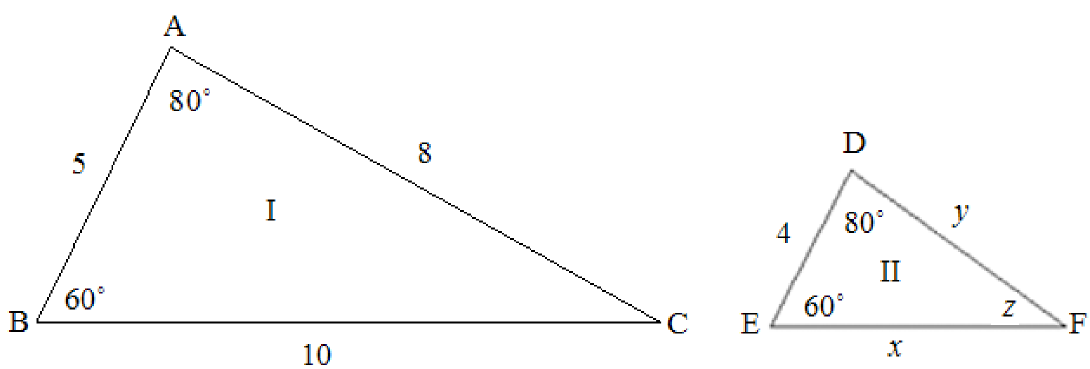
4) Congruent Triangles

Problem: Triangles I and II are congruent. Name the congruent triangles so that the vertices "correspond" and determine the values *x*, *y*, *u*, and *v*.



5) Similar Triangles

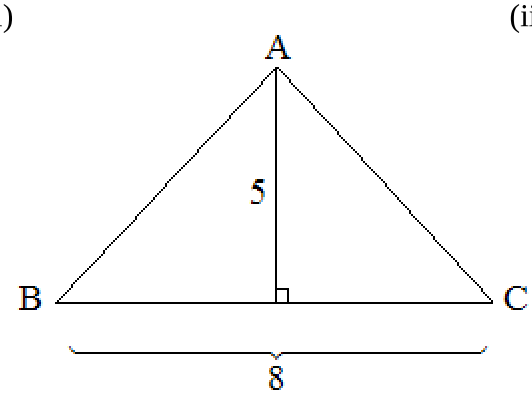
Problem: Triangles I and II are similar. Name the similar triangles so that the vertices "correspond" and determine the values *x*, *y*, and *z*.



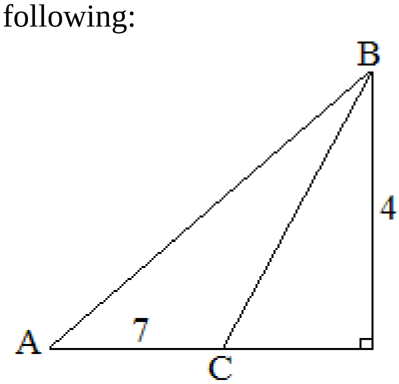
6) Area of a Triangle

Problem: Find the area of ΔABC in each of the following:

(i)



(ii)

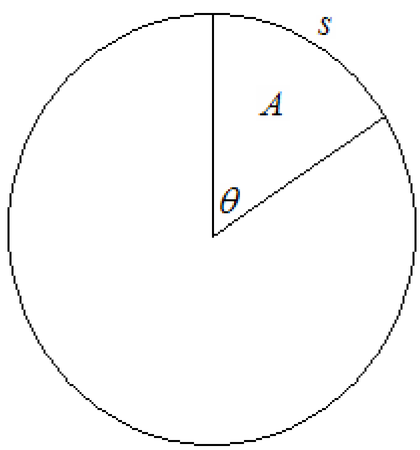


7) Area and Circumference of a Circle

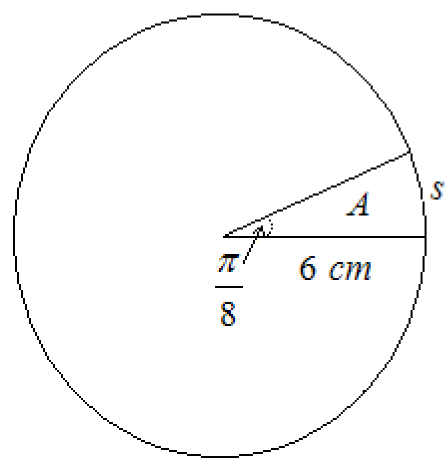
Problem: (i) Find the circumference and area of a circle of radius 8 cm.
(ii) Find the circumference and area of a circle of diameter 8 m.
(iii) If the area of a circle is 16π, find the radius.

8) Arc Length and Area of a Sector of a Circle

Problem: (i) State the arc length *s* and the area of the sector of this circle. Assume θ is in **radian** measure.



(ii) In the circle below, find the arc length *s* and the area *A*.

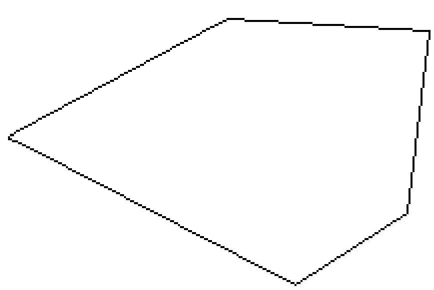


9) Volume of a Sphere, Box, Cone, Cylinder

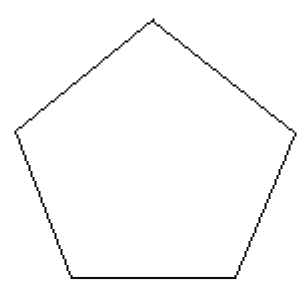
Problem: Find the volume V of
(i) a sphere of radius r = 3 cm
(ii) a rectangular box with length l = 8 cm, width w = 5 cm, and height h = 50 cm
(iii) a right circular cone with height h = 3 cm and base radius r = 0.04 m
(iv) a circular cylinder with height h = 0.2 m and radius
r = 4 cm.

10) Angles of a Polygon

Problem: i) Find the sum of the interior angles of a pentagon.



(ii) If all the interior angles of a pentagon are equal, how much is each interior angle?



Part 5: Basic Graphs

1) Graphing $y = x^n$, $n \in \mathbb{N}$

Problem: (i) Graph the curves $y = x^2$ and $y = x^4$ on the same set of axes.
(ii) Graph the curves $y = x^3$ and $y = x^5$ on the same set of axes.

2) Graphing $y = x^{m/n}$, $m, n \in \mathbb{N}$, n Odd, and m/n is a Reduced Fraction

Problem: (i) Graph the curves $y = x^{1/3}$ and $y = x^{2/3}$ on the same set of axes.
(ii) Graph the curves $y = x^{4/3}$ and $y = x^{5/3}$ on the same set of axes.

3) Graphing $y = x^{m/n}$, $m, n \in \mathbb{N}$, n Even, and m/n is a Reduced Fraction

Problem: Graph the curves $y = x^{1/2}$ and $y = x^{3/2}$ on the same set of axes.

4) Graphing $y = x^{-n} = \dfrac{1}{x^n}$, $n \in \mathbb{N}$

Problem: Graph the curves $y = x^{-1} = \dfrac{1}{x}$ and $y = x^{-2} = \dfrac{1}{x^2}$ on the same set of axes.

5) Graphing $y = x^{-\frac{1}{n}} = \dfrac{1}{x^{\frac{1}{n}}}$, $n \in \mathbb{N}$

Problem: Graph the curves $y = \dfrac{1}{x^{1/2}}$ and $y = \dfrac{1}{x^{1/3}}$ on the same set of axes.

6) Transformations (New Graphs from a Given Graph)

Problem: Given $y = f(x) = x^2$, graph and describe each of the following relative to *f*:
(i) $y = f(x) + 2$ (ii) $y = f(x) - 2$ (iii) $y = f(x+2)$ (iv) $y = f(x-2)$
(v) $y = f(2x)$ (vi) $y = 2f(x)$

Part 6: Solving Equations and Inequalities

1) Solving Linear Equations in One Variable

Problem: Solve each of the following equations:

(i) $4x + 20 = 2 - 5x$ (ii) $8(x-4) + x = 6(x-5) - (1-x)$ (iii) $\dfrac{x}{3} - \dfrac{2x}{5} + \dfrac{1}{30} = \dfrac{7}{10}$

2) Solving Linear Inequalities

Problem: Solve the following inequalities:
(i) $4x - 5 \le 2x + 9$ (ii) $2x + 7 > 5x - 1$ (iii) $x - 4 < x + 6$

3) Solving Two Linear Equations in Two Variables

Problem: Solve for *x* and *y*: (E1 and E2 refer to equation 1 and equation 2.)

(i) E1: $x + 2y = -1$	(ii) E1: $x - 2y = 6$	(iii) E1: $x - 2y = 6$
E2: $5x - 2y = 7$	E2: $3x - 6y = 18$	E2: $3x - 6y = 3$

4) Solving Quadratic Equations

Problem: Solve the following quadratic equations:
(i) $x^2 - 5x + 6 = 0$ (ii) $3x^2 - 7x + 2 = 0$

5) Solving Equations Involving Square Roots

Problem: Solve the following equations for *x*:

(i) $\sqrt{x-2} = 5$ (ii) $\sqrt{4-3x} = x + 12$ (iii) $\sqrt{1+2x} - \sqrt{x} = 1$

Part 7: Graphing Second Order Relations

1) The Parabola

Problem: Graph the following parabolas and identify the vertex and the axis of symmetry: (i) $y = x^2$ (ii) $y = 2(x+1)^2 - 3$

2) The Circle

Problem: Graph the following circles and identify the radius and the centre:
(i) $x^2 + y^2 = 4$ (ii) $(x-1)^2 + (y+2)^2 = 9$

3) The Ellipse

Problem: Graph the following ellipses and identify the centre and the major and minor axes: (i) $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (ii) $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$

4) The Hyperbola

Problem: Graph the following hyperbolas and identify the centre and intercepts:
(i) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ (ii) $\frac{y^2}{9} - \frac{x^2}{4} = 1$

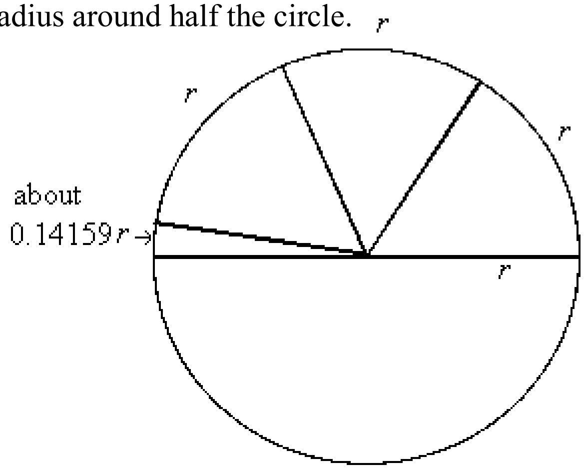
Part 8: Trigonometry

1) Angles in Standard Position: Degree Measure

Problem: (i) Draw in standard position the following angles:
(a) 30° (b) 225° (c) -80° (d) -190° (e) 390°

2) The Meaning of π

Problem: In the circle below with radius *r*, you can "fit" three and a "little portion more" of a radius around half the circle.



We give a name to this number of radii. (i) The name is ______.

(ii) So the length of a half circle is given by _______.

(iii) This is why the circumference is given by ______.

3) Angles in Standard Position: Radian Measure

Problems: 1) Draw in standard position the following angles:
(i) $\dfrac{\pi}{6}$ (ii) $\dfrac{5\pi}{4}$ (iii) $-\dfrac{4\pi}{9}$.
2) Give all angles, in radians, which are "co-terminal" with $\pi/6$ radians.

4) Degrees to Radians

Problem: Express each of the following in radian measure:

(i) 25° (ii) -150° (iii) 1060° (Remember 180 degrees = π radians.)

5) Radians to Degrees

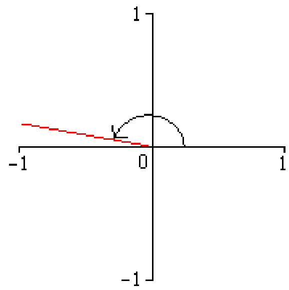
Problem: Express each of the following radian measures in degrees:

(i) $\frac{4\pi}{9}$ (ii) $\frac{2}{5}$ (iii) $-\frac{7\pi}{6}$ (iv) $-\frac{4\pi}{3}$ (Remember π radians = 180 degrees)

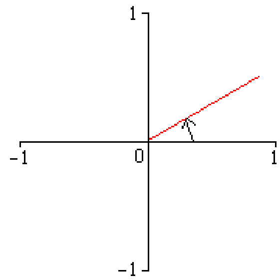
6) Relating Angles in Standard Position in Quadrants One and Two

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

Problem: (i) Below, the second quadrant angle 170° is drawn in standard position. Find and illustrate the related first quadrant angle, using the interval $\left(0, 90^\circ\right)$.



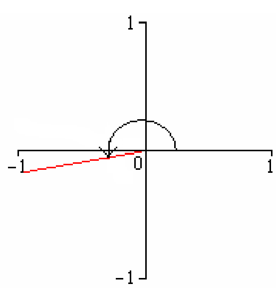
(ii) Below, the first quadrant angle $\frac{\pi}{6}$ is drawn in standard position. Find and illustrate the related second quadrant angle, using the interval $\left(\frac{\pi}{2}, \pi\right)$.



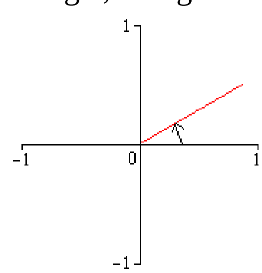
7) Relating Angles in Standard Position in Quadrants One and Three

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

Problem: (i) Below, the third quadrant angle 190° is drawn in standard position. Find and illustrate its related first quadrant angle, using the interval $\left(0, 90^\circ\right)$.



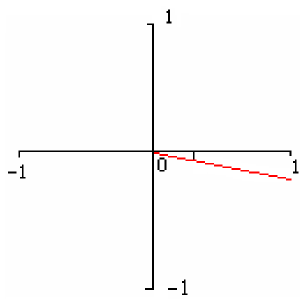
(ii) Below, the first quadrant angle $\frac{\pi}{6}$ is drawn in standard position. Find and illustrate its related third quadrant angle, using the interval $\left(\pi, \frac{3\pi}{2}\right)$.



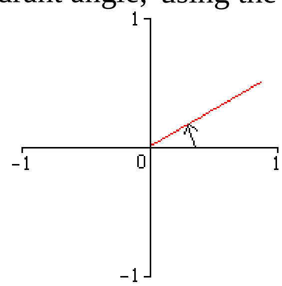
8) Relating Angles in Standard Position in Quadrants One and Four

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

Problem: (i) Below, the fourth quadrant angle -10° is drawn in standard position. Find and illustrate its related first quadrant angle, using the interval $\left(0,90^\circ\right)$.



(ii) Below, the first quadrant angle $\frac{\pi}{6}$ is drawn in standard position. Find and illustrate its related fourth quadrant angle, using the interval $\left(-\frac{\pi}{2},0\right)$.



9) Relating an Angle in Standard Position to its "Relatives" in the other Quadrants

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

For this question, use the following restrictions:

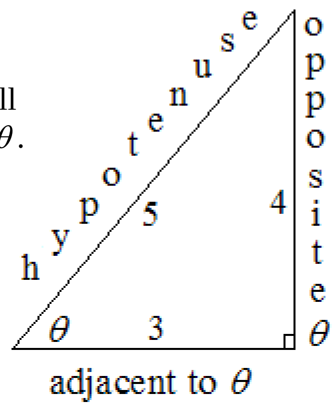
Quadrant	Angle
1	Degrees: $0^\circ < \theta < 90^\circ$ or Radians: $0 < \theta < \frac{\pi}{2}$
2	Degrees: $90^\circ < \theta < 180^\circ$ or Radians: $\frac{\pi}{2} < \theta < \pi$
3	Degrees: $180^\circ < \theta < 270^\circ$ or Radians: $\pi < \theta < \frac{3\pi}{2}$
4	Degrees: $-90^\circ < \theta < 0^\circ$ or Radians: $-\frac{\pi}{2} < \theta < 0$

Problem: State the standard position "relatives" of
(i) 50° (ii) 170° (iii) 250° (iv) -60° in each of the other quadrants.

10) Trigonometric Ratios in Right Triangles: SOHCAHTOA

Problem:

From the triangle, identify all six trigonometric ratios for θ.

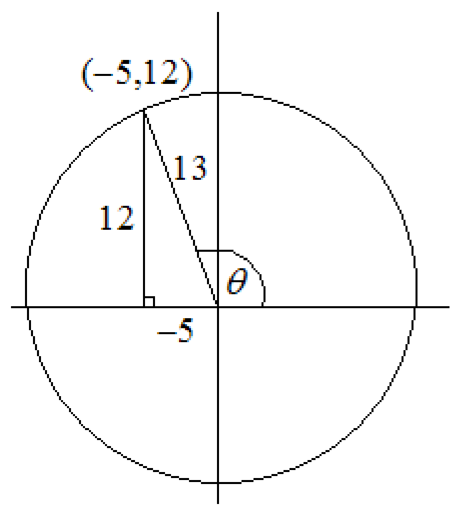


11) Trigonometric Ratios Using the Circle: Part I

Problem: Let θ be an angle which is **not** between 0° and 90°.
By drawing the angle in standard position and letting it puncture the unit circle $x^2 + y^2 = 1$ at a point (x, y), find the sin, cos, and tan of θ.

12) Trigonometric Ratios Using the Circle: Part II

Problem: In the diagram, θ, where $90^\circ < \theta < 180^\circ$, is a second quadrant angle in standard position whose terminal side punctures the circle (centered at the origin, with radius 13) at the point P(−5,12). State the six trigonometric ratios of θ.



13) Trigonometric Ratios for the 45°, 45°, 90° Triangle

Problem: Find the values of all the six trigonometric ratios of $45^\circ = \frac{\pi}{4}$.

14) Trigonometric Ratios for the 30°, 60°, 90° Triangle

Problem: Find the values of all the six trigonometric ratios of $60^\circ = \frac{\pi}{3}$ and $30^\circ = \frac{\pi}{6}$.

15) Trigonometric Ratios for the 0°, 90°, 180°, 270°;
ie., Trig Ratios For Angles on the Axes

First please re-read "11) Trigonometric Ratios Using the Circle: Part I".

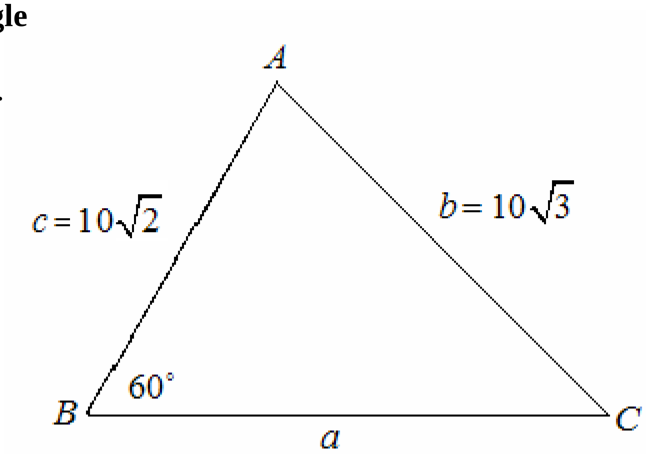
Problem: Find the six trig ratios for $0^\circ = 0$ radians and $90^\circ = \dfrac{\pi}{2}$ radians.

16) CAST RULE

Problems: 1) Given $\tan(\theta) = -4/3$, find the values of $\sin(\theta)$ and $\cos(\theta)$ if
(i) θ is a second quadrant angle.
(ii) θ is a fourth quadrant angle.
2) Why can't θ be a first or third quadrant angle?

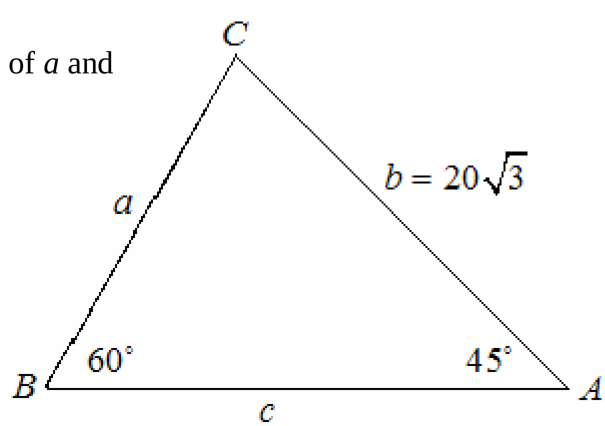
17) Sine Law: Find an Angle

Problem: Find $\angle C$ and $\angle$A.



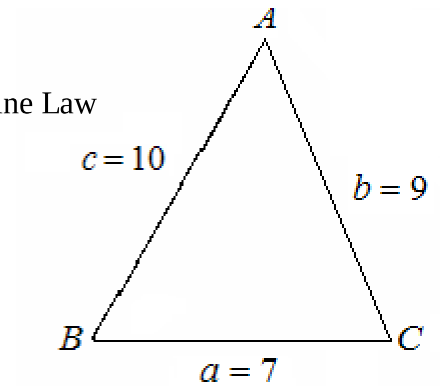
18) Sine Law: Find a Side

Problem: Find (i) the exact value of *a* and (ii) *c* accurate to two decimals.



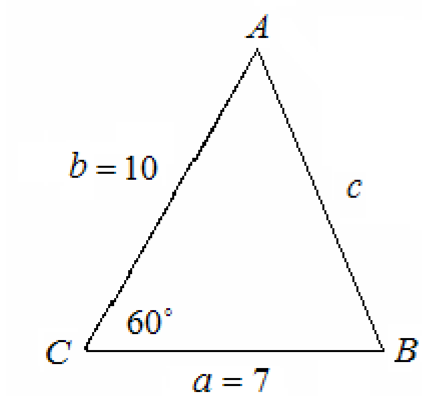
19) Cosine Law: Find an angle

Problem: In ΔABC, use the Cosine Law to find $\angle B$ to the nearest degree.



20) Cosine Law: Find a Side

Problem: In ΔABC, use the Cosine Law to find $c = AB$ correct to two decimal places.



21) The Graphs of the Sin, Cos, and Tan Functions

Problem: Graph for $0 \le \theta \le 2\pi$: (i) $y = \sin(\theta)$ (ii) $y = \cos(\theta)$ (iii) $y = \tan(\theta)$

22) Period of the Sin, Cos, and Tan Functions

Problems: State the period of each of the following:

1) (i) $y = \sin(x)$ (ii) $y = \sin(2x)$ (iii) $y = \sin\left(\frac{x}{2}\right)$

2) (i) $y = \cos(x)$ (ii) $y = \cos(3x)$ (iii) $y = \cos\left(\frac{x}{3}\right)$

3) (i) $y = \tan(x)$ (ii) $y = \tan(4x)$ (iii) $y = \tan\left(\frac{\pi}{2}x\right)$

23) The Graphs of the Csc, Sec, and Cot Functions

Problem: Graph for $0 \le \theta \le 2\pi$: (i) $y = \csc(\theta)$ (ii) $y = \sec(\theta)$ (iii) $y = \cot(\theta)$

24) Trig Formulas That You Should Know

Problem: Complete the following formulas:

(i) $\sin^2(\theta) + \cos^2(\theta) = \boxed{}$; $1 + \tan^2(\theta) = \boxed{}$

(ii) $\sin(-\theta) = \boxed{}$; $\cos(-\theta) = \boxed{}$

(iii) $\sin\left(\frac{\pi}{2} - \theta\right) = \boxed{}$; $\cos\left(\frac{\pi}{2} - \theta\right) = \boxed{}$

(iv) $\sin(A \pm B) = \boxed{}$; $\cos(A \pm B) = \boxed{}$

(v) $\sin(2A) = \boxed{}$; $\cos(2A) = \boxed{}$

Part 9: Exponents and Logarithms

1) Exponents

Problems: 1) Evaluate: (i) 2^3 (ii) $\left(\frac{3}{5}\right)^3$ (iii) 4^{-2} (iv) 10^0 (v) $\left(\frac{1}{0.01}\right)^{-3}$

2) Simplify: (i) $\frac{x^5x^4}{x^7}$ (ii) w^{-1} (iii) $\frac{1}{w^{-3}}$ (iv) $\left(z^{\frac{2}{3}}\right)^{10}$ (v) $\left(\frac{a^7b^3}{c^2}\right)^5$

2) Logarithms (Log means FIND THE EXPONENT!)

Problems: 1) Evaluate: (i) $\log_2 8$ (ii) $\log_2\left(\frac{1}{8}\right)$ (iii) $\log_3 1$ (iv) $\log_5 5$

(v) $\log 10000$ (vi) $\ln(e^7)$*

2) Expand using log properties: $\ln\left(\frac{x^3y^{1/2}}{z^4}\right)$

3) Change $\log_5 7$ to log with base 3, then with base 10, and finally with base *e.*

**"e" and "ln" refer to the "natural logarithm". If you have not taken calculus, you may be totally unfamiliar with e. If so, treat it as a constant just as you would, for example, a.*

3) Exponential Graphs

Problem: (i) Graph the exponential functions $y = 2^x$ and $y = 3^x$ on the same set of axes.

(ii) Graph the exponential functions $y = 2^{-x} = \frac{1}{2^x}$ and $y = 3^{-x} = \frac{1}{3^x}$ on the same set of axes.

4) Logarithmic Graphs

Problem: Graph the functions $y = \log_2(x)$ and $y = \log_3(x)$ on the same set of axes.

5) Exponents to Logarithms and Vice-Versa

Problems: 1) Change to a log equation: (i) $32=2^5$ (ii) $y=10^x$ (iii) $y=4x^k$ (Use base 10.)
2) Change to an exponential equation: (i) $\log_3 81=4$ (ii) $y=\log_5 x$

6) Using a Calculator to Evaluate Exponents and Logs

Problems: Use a calculator to give answers rounded to two decimal places.
1) (i) $2^{3.3}$ (ii) $5^{1/7}$ (iii) $(-10)^{1/3}$
2) (i) $\log(25)$ (ii) $\ln(25)*$ (iii) $\log_2(25)$

**"e" and "ln" refer to the "natural logarithm". If you have not taken calculus, you may be totally unfamiliar with e. If so, treat it as a constant just as you would, for example, a.*

Part 10: Calculus

1) Limits

Problem: Evaluate the following limits:

(i) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^3-27}$ (ii) $\lim\limits_{x\to -4^+}\dfrac{x^2+5}{x^2+x-12}$ (iii) $\lim\limits_{\theta\to 0}\dfrac{2\theta}{\sin(3\theta)}$ (iv) $\lim\limits_{x\to\infty}\dfrac{3x+5}{4x-7}$

2) The Derivative from First Principles (that is, from the Definition of the Derivative)

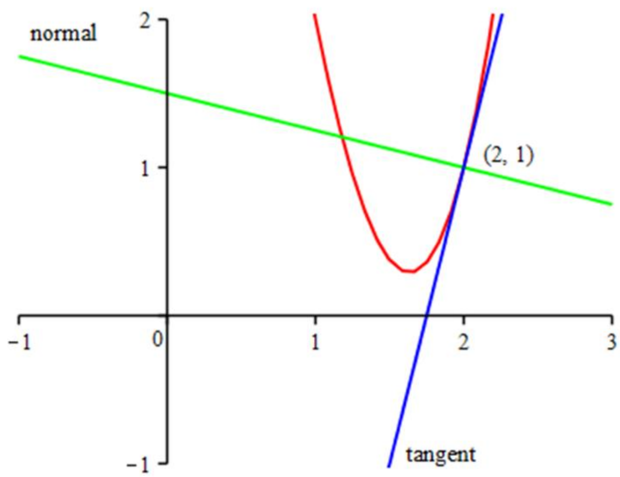
Problem: Find the derivative $\dfrac{dy}{dx}$ from first principles, where $y=2x^2-3x+1$.

3) Using the Derivative Rules

Problem: Use the derivative rules to find $\dfrac{dy}{dx}$, $\dfrac{ds}{dt}$, and $\dfrac{dw}{dz}$. Do not simplify your answers. (i) $y=(\pi x+3x^4)\sin(x)$ (ii) $s=\dfrac{\cos(t)-t}{e^t+et}$ (iii) $w=\left(\sin(z^{\frac{1}{3}})+z^{-\frac{1}{3}}\right)^5$

4) Finding the Equations of the Tangent and Normal to a Curve at a Given Point

Problem: Find the tangent and normal to $y=x^3-8x+9$ at the point where $x=2$.



5) Exponential Growth and Decay

Problem: A city had a population of 2 million in 2000 and 2.5 million in 2010. The exponential growth model for the population in millions is $P(t) = P_0 e^{kt}$. Assume $t = 0$ corresponds to the year 2000.
(i) Find P_0 and k. (ii) Predict the population in the year 2030 (iii) What is the rate of population growth in 2030?

6) Vertical and Horizontal Asymptotes

Problem: Find the vertical and horizontal asymptotes of $y = \dfrac{x^2}{x^2 - 4}$.

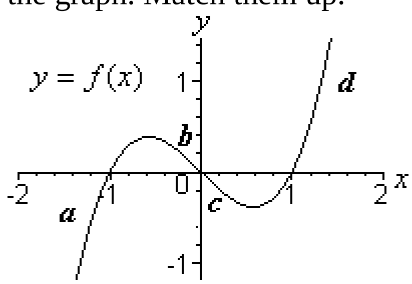
7) Using the Derivative to Determine Where a Function is Increasing and Where it is Decreasing

Problem: If $\dfrac{dy}{dx} = \dfrac{x-3}{(x+2)\left(x+1\right)}$, determine the intervals on which y is increasing and those on which it is decreasing.

8) Using the Second Derivative to Determine where a Function is Concave Up and where it is Concave Down

Problem: If $\dfrac{d^2y}{dx^2} = \dfrac{x-3}{(x+2)\left(x+1\right)}$, determine the intervals on which y is concave up and those on which it is concave down.

9) Using the First and Second Derivatives to Determine the Shape of a Function
Problem: Each y', y'' box below matches the conditions for one of the four shapes a, b, c, d on the graph. Match them up.



$y' > 0$ $y'' > 0$	$\square$	$y' > 0$ $y'' < 0$	$\square$	$y' < 0$ $y'' > 0$	$\square$	$y' < 0$ $y'' < 0$	$\square$

10) Optimization

Problem: Find the minimum distance between the point $(-1, -1)$ and the line $y = x + 6$.

Part 11: Applications and Extras

1) Applications of Sinusoidal Functions

Problem: Max is on a Ferris Wheel that has a diameter of 20 metres. He reaches the lowest part of the ride 5 seconds after getting on, and the highest part at 45 seconds so that a complete rotation takes 80 seconds. The lowest part of the ride is 3 metres above the ground. Determine a cosine function that models his ride on the Ferris Wheel.

2) Solving Word Problems

Problem: A cookie recipe has three times the amount of flour as it does butter and a bowl comfortably fits 8 cups. If the other ingredients take up ¼ of the mix, how much flour and how much butter should you combine to comfortably fit in the bowl?

3) Length and Midpoint of the Line Joining Two Points and the Point of Intersection of Two Lines

Problem:
(i) Find the length and midpoint of the line that connects the points $(12,-10)$ and $(15,-3)$.
(ii) What is the point of intersection between the lines $y = 2x + 3$ and $3y = 12x + 27$?

4) Optimization

Problem: Hannah is constructing a rectangular garden with a fence around it. She has 32 metres of fencing to put along the edges and she will need to leave a 2 metre gap for the gate. What dimensions should be used to maximize the area of the garden?

5) Mapping Points in a Transformation

Problem: For each transformation, identify the parent function. Then in (i), find the images of points $(-2,4)$ and $(4,16)$ on the graph of e under the transformation f. In (ii), find the images of points (10,0.1), (0.5, 2) on the graph of g under the transformation h. Use mapping notation.

(i) $e(x) \to f(x) = -\left(\frac{x}{3}+6\right)^2 + 4$ (ii) $g(x) \to h(x) = \frac{2}{x+1}$

6) Inverse functions

Problem: Find $f^{-1}(x)$ for each of the following and state the new Domain and Range.

(i) $f(x) = \sqrt[3]{x} + 19$ (ii) $f(x) = \frac{1}{x+3}$

7) Application of Quadratic Functions

Problem: A ball is thrown along a path represented by $h(t) = -5t^2 + 40t + 2$ where $h(t)$ is the height in metres above the ground and t is time in seconds, $t \geq 0$.
(i) What is the maximum height that the ball reaches? (ii) At what time does the ball hit the ground? (iii) What is the initial height of the ball?

8) Simple and Compound Interest

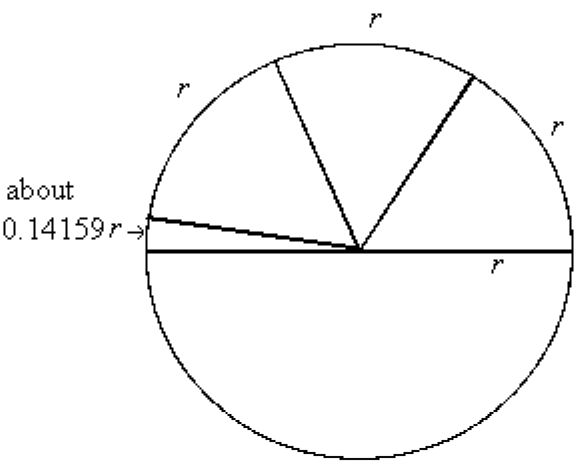
Problem: (i) You invest $1000 into an account that earns 6% interest compounded monthly. Determine the interest earned after 10 years.
(ii) How much interest would you earn if you were only receiving simple interest each month? (Simple Interest: $I = Prt$)

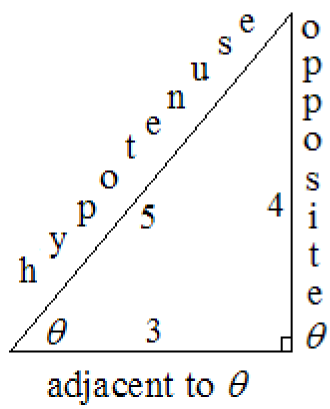
9) Mean, Median, and Mode

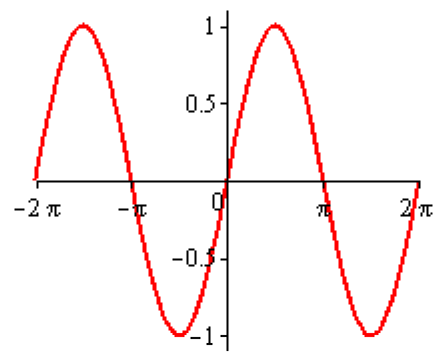
Problem: Find the mean, median, and mode of the following sets of data:
(i) 5.5, 2.1,7.4, 10.0, 7.3, 2.1, 9.8 (ii) 2, -3, 4, -6, 9, 1, -1, 10, -10, 12

Mathematics Post- Secondary Preparation Package 2020 (MP^3)

SOLUTIONS



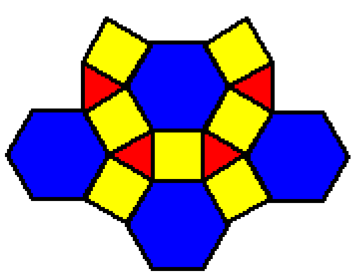


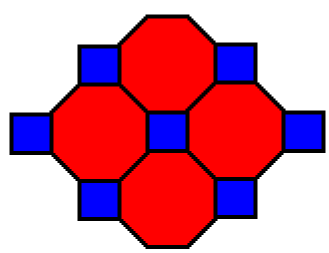


(i) $\cot^2(\theta)+1=\boxed{}$ (ii) $\tan(-\theta)=\boxed{}$ (iii) $\tan\left(\frac{\pi}{2}-\theta\right)=\boxed{}$

(iv) $\tan(A\pm B)=\boxed{}$ (v) $\tan(2A)=\boxed{}$

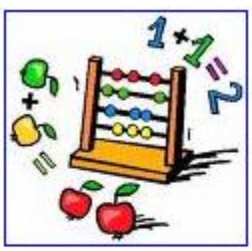
$$ax^2+bx+c=0 \Leftrightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$





Solutions Part 1

Brushing up on



Numerical Skills

1) Adding and Subtracting Fractions

Problem: Evaluate **without a calculator!**

(i) $\frac{2}{3}+2\frac{5}{6}$ (ii) $3\frac{2}{5}-2\frac{3}{4}$ (iii) $\frac{1}{6a}+\frac{2}{3a}-\frac{5}{12a}$

Solution:

(i) $\frac{2}{3}+2\frac{5}{6}=\frac{4}{6}+\frac{17}{6}=\frac{21}{6}=\frac{7}{2}\overset{\boxed{\text{optional}}}{=}3\frac{1}{2}$

(ii) $3\frac{2}{5}-2\frac{3}{4}=\frac{68}{20}-\frac{55}{20}=\frac{13}{20}$ or $3\frac{2}{5}-2\frac{3}{4}=1\frac{2}{5}-\frac{3}{4}=\frac{28}{20}-\frac{15}{20}=\frac{13}{20}$

(iii) $\frac{1}{6a}+\frac{2}{3a}-\frac{5}{12a}=\frac{2}{12a}+\frac{8}{12a}-\frac{5}{12a}=\frac{5}{12a}$

Notes: Use the "lowest" common denominator.

Common error: Not using the lowest common denominator leads to larger numerators and more possibility of making mechanical errors.

Practice: (i) $\frac{3}{4}+1\frac{1}{2}$ (ii) $3\frac{1}{7}-1\frac{2}{3}$ (iii) $\frac{3}{5}+\frac{2}{4}-\frac{1}{30}$

Answer: (i) $\frac{9}{4}=2\frac{1}{4}$ (ii) $\frac{31}{21}=1\frac{10}{21}$ (iii) $\frac{16}{15}$

A Great Video for More Detail:
https://www.khanacademy.org/math/arithmetic/fraction-arithmetic/arith-review-add-sub-fractions/v/adding-small-fractions-with-unlike-denominators
A Great Website for More Detail: http://www.math.com/

2) Multiplying and Dividing Fractions

Problem: Evaluate **without a calculator!**

(i) $\frac{5}{3}\times\frac{12}{25}$ (ii) $1\frac{2}{5}\times\frac{3}{4}$ (iii) $3\times\frac{4}{5}$ (iv) $\frac{\left(\frac{5}{3}\right)}{\left(\frac{25}{12}\right)}$ (v) $\frac{\left(\frac{4}{3}\right)}{5}$ (vi) $\frac{4}{\left(\frac{3}{5}\right)}$

Solution:

(i) $\frac{\overset{1}{5}}{\underset{1}{\cancel{3}}}\times\frac{\overset{4}{\cancel{12}}}{\underset{5}{\cancel{25}}}=\frac{1\times 4}{1\times 5}=\frac{4}{5}$ (ii) $1\frac{2}{5}\times\frac{3}{4}=\frac{7}{5}\times\frac{3}{4}=\frac{21}{20}\overset{\boxed{\text{optional}}}{=}1\frac{1}{20}$

(iii) $3\times\frac{4}{5}=\frac{3}{1}\times\frac{4}{5}=\frac{12}{5}=2\frac{2}{5}$ (iv) $\frac{\left(\frac{5}{3}\right)}{\left(\frac{25}{12}\right)}\overset{\boxed{\text{Invert and multiply.}}}{=}\frac{\overset{1}{5}}{\underset{1}{\cancel{3}}}\times\frac{\overset{4}{\cancel{12}}}{\underset{5}{\cancel{25}}}=\frac{1\times 4}{1\times 5}=\frac{4}{5}$

(v) $\frac{\left(\frac{4}{3}\right)}{5}=\frac{\left(\frac{4}{3}\right)}{\left(\frac{5}{1}\right)}=\frac{4}{3}\times\frac{1}{5}=\frac{4}{15}$ (vi) $\frac{4}{\left(\frac{3}{5}\right)}=\frac{\left(\frac{4}{1}\right)}{\left(\frac{3}{5}\right)}=\frac{4}{1}\times\frac{5}{3}=\frac{20}{3}$

Note: Common denominators are irrelevant when multiplying and dividing.

Common error: $\frac{\left(\frac{a}{b}\right)}{c}=\frac{a}{b}\times\frac{c}{1}$

Practice: (i) $\frac{3}{4}\times\frac{8}{15}$ (ii) $\frac{\left(\frac{2}{7}\right)}{\left(\frac{12}{21}\right)}$ (iii) $\frac{\left(\frac{22}{3}\right)}{11}$ (iv) $\frac{22}{\left(\frac{3}{11}\right)}$

Answer: (i) $\frac{2}{5}$ (ii) $\frac{1}{2}$ (iii) $\frac{2}{3}$ (iv) $\frac{242}{3}$

A Great Video for More Detail: https://www.youtube.com/watch?v=T3D9z6lUldM
A Great Website for More Detail: http://www.math.com

3) Working with Decimals.

Problem: Evaluate **without a calculator!**

(i) $1.02 + .023$ (ii) $1.02 - 2.57$ (iii) $1.2 \times .05$ (iv) $\dfrac{4.291}{3}$

(v) $\dfrac{4.291}{0.3}$ (vi) $\dfrac{0.004291}{0.03}$

Solution:

Align the decimals.

(i) $1.02 + .023 = 1.043$ (ii) $1.02 - 2.57 = -(2.57 - 1.02) = -1.55$

Work out $12 \times 5 = 60$. Now start on the right and count THREE decimal places to the left.

(iii) $1.2 \times .05 = 0.06$ (iv) $\dfrac{4.291}{3} = 1.430\overline{3} = 1.4303333...$

Multiply the bottom by 10 to move the decimal after the 3. Multiply the top by 10 so we don't change the value.

(v) $\dfrac{4.291}{0.3} = \dfrac{4.291}{0.3} \times \dfrac{10}{10} = \dfrac{42.91}{3} = 14.30\overline{3}$

(vi) $\dfrac{0.004291}{0.03} = \dfrac{0.004291}{0.03} \times \dfrac{100}{100} = \dfrac{.4291}{3} = 0.1430\overline{3}$

Note: When dividing by a decimal, most of us were taught to move the decimal the same number of places to the right in questions such as (iv) and (v). What we are really doing is multiplying the numerator and denominator by the same power of 10.

Common error: Running to our calculators when there is a decimal in the denominator because division by a decimal is scary.

Practice: (i) $21.021 - 25.555$ (ii) 30×0.00005 (iii) $\dfrac{3.6}{.000018}$

Answer: (i) -4.534 (ii) 0.0015 (iii) 200000

A Great Video for More Detail: https://www.youtube.com/watch?v=UCBXoLb2ItI
A Great Website for More Detail: http://www.math.com

4) Roots and Radicals

Problems: 1) Write as simplified mixed radicals:

(i) $\sqrt{40}$ (ii) $2\sqrt{27}$ (iii) $\sqrt{x^4y^7}$ (iv) $\sqrt[3]{x^4y^7}$

2) Write as entire radicals: (i) $3\sqrt{2}$ (ii) $\frac{4}{9}$ (iii) $xy^4\sqrt{xy}$ (iv) $xy^4\sqrt[3]{xy}$

3) Evaluate: (i) $\sqrt{121}$ (ii) $\left(\frac{27}{64}\right)^{2/3}$ (iii) $32^{1/5}$ (iv) $(-32)^{1/5}$ (v) $(-64)^{1/6}$

Solution:

1)(i) $2\sqrt{10}$ (ii) $6\sqrt{3}$ (iii) $x^2y^3\sqrt{y}$ (iv) $\sqrt[3]{x^4y^7} = xy^2\sqrt[3]{xy}$

2) (i) $\sqrt{18}$ (ii) $\sqrt{\frac{16}{81}}$ (iii) $\sqrt{x^3y^9}$ (iv) $\sqrt[3]{x^4y^{13}}$

3) (i) 11 (ii) $\frac{9}{16}$ (iii) 2 (iv) -2 (v) does not exist (as a “real number”)

Note: Radical signs are just special cases of the more common exponents signs. For example, $\sqrt{x^3} = x^{\frac{3}{2}}$.

Common error: In real numbers, we can not have the square root or fourth root or sixth root, etc., of a negative number. We CAN have the third root or the fifth root or the seventh root, etc., of a negative number. By the way, many calculators give an error message when ask for the odd root of a negative. Boo to the calculator’s programmer!

Practice: 1) Write as mixed radicals: (i) $\sqrt{1000}$ (ii) $2\sqrt{27}$ (iii) $\sqrt{x^{12}y^9}$ (iv) $\sqrt[5]{x^{15}y^7}$

2) Write as entire radicals: (i) $5\sqrt{5}$ (ii) $\frac{4}{9}$ (iii) $xy^5\sqrt{x^{1/3}y}$ (iv) $x^2y^4\sqrt[4]{x}$

3) Evaluate: (i) $\sqrt[3]{-27}$ (ii) $\left(\frac{36}{25}\right)^{3/2}$ (iii) $(-243)^{1/5}$ (iv) $(-81)^{3/4}$

Answers: 1) (i) $10\sqrt{10}$ (ii) $6\sqrt{3}$ (iii) $x^6y^4\sqrt{y}$ (iv) $x^3y\sqrt[5]{y^2}$

2) (i) $\sqrt{125}$ (ii) $\sqrt{\frac{16}{81}}$ (iii) $\sqrt{x^{7/3}y^{11}}$ (iv) $\sqrt[4]{x^9y^{16}}$

3) (i) -3 (ii) $\frac{216}{125}$ (iii) -3 (iv) does not exist

A Great Video for More Detail: https://www.youtube.com/watch?v=qpqeyzrEP0o
A Great Website for More Detail:
https://www.wyzant.com/resources/lessons/math/algebra/square_roots_and_radicals

5) Absolute Value

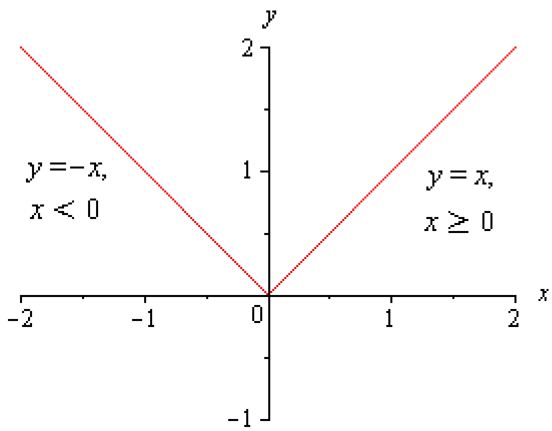
Problems: 1) Evaluate (i) $|10|$ (ii) $|-10|$ (iii) $|0|$

2) Write $|x|$ without absolute value bars if (i) $x > 0$ (ii) $x < 0$

3) Draw the graph of $y = |x|$

Solution: 1)(i) 10 (ii) 10 (iii) 0 2)(i) $|x| = x$ (ii) $|x| = -x$

3)



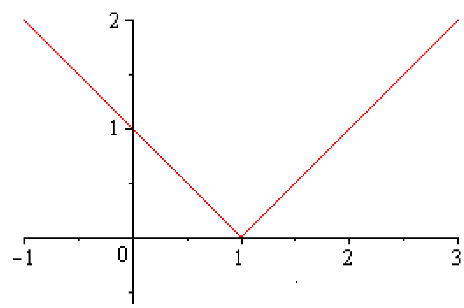
Note: When, for $x < 0$, we write $|x| = -x$, remember there is a negative **inside the *x*!**

Common error: Assuming $|-x| = x$. This may or may not be true **depending on *x*.** For example, if $x = 3$, then $|-x| = |-3| = 3 = x$. But if $x = -3$, then $|-x| = |-(-3)| = |3| = 3 \neq x$; in fact, here $|-x| = -x$. It depends on x.

Practice: 1) Evaluate (i) $|-.001|$ (ii) $|.001|$ (iii) $|-0|$
2) Write $y = |x-1|$ without using absolute value notation and graph the function.

Answers: 1) (i) .001 (ii) .001 (iii) 0

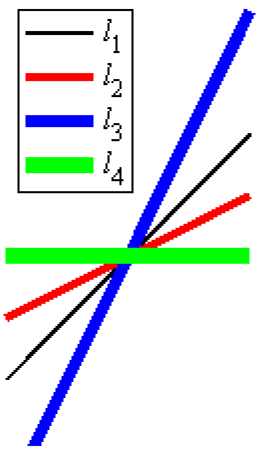
2) $y = \begin{cases} -(x-1), & \text{if } x < 1 \\ x-1, & \text{if } x \geq 1 \end{cases}$



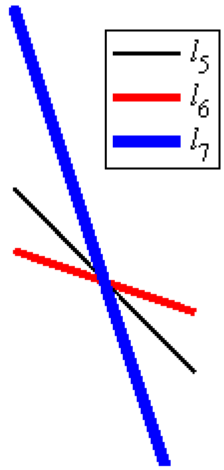
A Great Video for More Detail: https://www.youtube.com/watch?v=u6zDpUL5RkU
A Great Website for More Detail: **https://www.mathsisfun.com/numbers/absolute-value.html**

Solutions Part 2

Lines



and



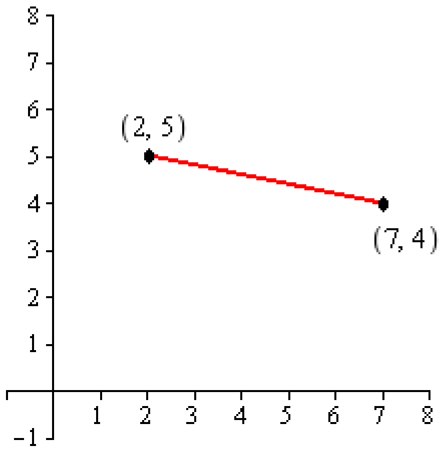
Slopes

1) Finding the Slope of a Line

Problem: Find the slope of the line joining (2,5) to (7,4).

Solution:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-5}{7-2} = -\frac{1}{5}$$



Note: It doesn't matter which point you use as (x_1, y_1).

Common error: $\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_1 - x_2} = \frac{4-5}{2-7} = \frac{1}{5}$

Practice: Find the slope of the line joining (−2,8) to (4,−5).

Answer: $\text{slope} = -\frac{13}{6}$

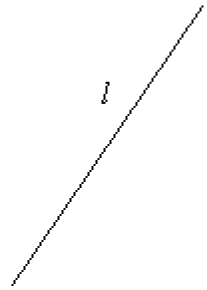
A Great Video for More Detail: https://www.youtube.com/watch?v=wvzBH46D6ho
A Great Website for More Detail: purplemath.com/modules/strtlneq.htm

2) Parallel and Perpendicular Slopes

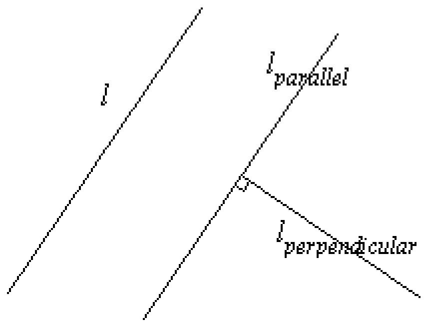
Problem: Find the slope of the line

(i) parallel (ii) perpendicular

to a line *l* with slope 2.



Solution: (i) Parallel slope = 2 (ii) Perpendicular slope $= -\frac{1}{2}$.



Note: A line with slope 0 is horizontal. In this case, the perpendicular line, which is vertical, has undefined (or infinite) slope. Apart from the case of horizontal/vertical lines, the slope of a line perpendicular to a line with slope m satisfies $m_{perpendicular} = -\frac{1}{m}$, that is, these slopes are "negative reciprocals".

Common error: Perpendicular slope $= \frac{1}{2}$.

Practice: Find the slope of a line (i) parallel (ii) perpendicular to a line with slope $-\frac{3}{2}$.

Answer: (i) $-\frac{3}{2}$ (ii) $\frac{2}{3}$

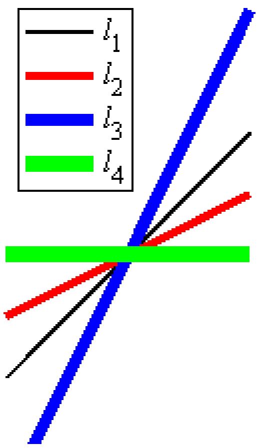
A Great Video for More Detail: https://www.youtube.com/watch?v=9hryH94KFJA
A Great Website for More Detail: https://www.mathsisfun.com/algebra/line-parallel-perpendicular.html

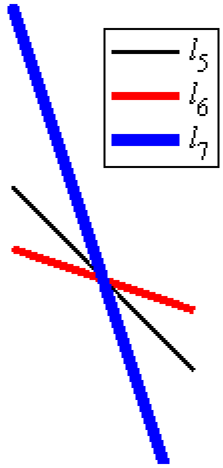
3) Interpreting Slope

Problem:

(i) Match the slopes $0, \frac{1}{2}, 1, 2$ with the lines l_1, l_2, l_3, l_4.

(ii) Match the slopes $-\frac{1}{3}, -1, -3$ with the lines l_5, l_6, l_7.





Solution: (i)

line	l_1	l_2	l_3	l_4
slope	1	$\frac{1}{2}$	2	0

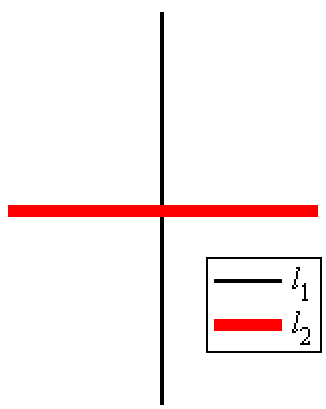
(ii)

line	l_5	l_6	l_7
slope	-1	$-\frac{1}{3}$	-3

Notes: For positive slope, as x increases, y increases. For negative slope, as x increases, y decreases. As you walk **to the right**, with positive slope you are walking uphill. With negative slope, you are walking downhill.

Common error: Misidentifying slope because the axes are not scaled one to one.

Practice: Match the slopes 0 and undefined (or ∞) with the lines l_1 and l_2.



Answer: l_1 has undefined (or infinite) slope and l_2 has 0 slope.

A Great Video for More Detail: https://www.youtube.com/watch?v=zihsQC0IUd8
A Great Website for More Detail: https://courses.lumenlearning.com/economics2e-demo/chapter/interpreting-slope/

4) Finding the Slope and Intercepts From the Equation of a Line

Problem: Find the slope and x and y intercepts for each of the following lines:

(i) $y=-2x+5$ (ii) $6x+2y=1$ (iii) $y=5$ (iv) $x=1$

Solution: In the equation $y=mx+b$, m is the slope and b is the y intercept.

(i) $y=-2x+5$: $m=-2$ and $b=5$. To find the x intercept, set $-2x+5=0 \Rightarrow x=\frac{5}{2}$.

(ii) $6x+2y=1$: x intercept $\overset{\boxed{\text{Set } y=0.}}{\Rightarrow}$ $6x=1 \Rightarrow x=\frac{1}{6}$

y intercept $\overset{\boxed{\text{Set } x=0.}}{\Rightarrow}$ $2y=1 \Rightarrow y=\frac{1}{2}$

slope $\overset{\boxed{\text{Rewrite the equation in the form } y=mx+b.}}{\Rightarrow}$ $y=-3x+\frac{1}{2} \Rightarrow m=-3$

(iii) $y=5=0x+5$ This is a horizontal line, that is, a line parallel to the x axis. The slope m is 0, the y intercept is 5 and there is no x intercept.

(iv) $x=1$ This is a vertical line, that is, a line parallel to the y axis. The slope is infinite (or undefined). The x intercept is 1 and there is no y intercept.

Note: The y intercept corresponds to the point $(0, y)$ on the line and the x intercept corresponds to $(x,0)$.

Common error: Confusing the y intercept b with the point $(0,b)$.

Practice: Find the slope and x and y intercepts of the line $y=-2x+6$.

Answer: $m=-2$; x intercept $=3$; y intercept $=6$

A Great Video for More Detail: https://www.youtube.com/watch?v=7LJPSo4U-SM
A Great Website for More Detail:
https://www.mathwarehouse.com/algebra/linear_equation/slope-intercept-form.php

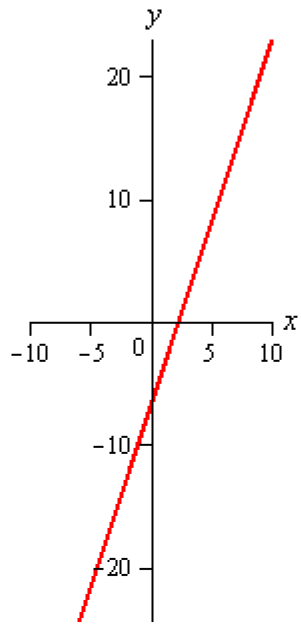
5) Finding the Equation of a Line Given Two Points

Problem: Find the equation of the line joining (4,5) to (7,14). Draw the graph.

Solution: The easiest formula for finding the equation of a line with slope m and point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{14-5}{7-4} = 3$ Use (4,5) as (x_1, y_1).

$\therefore\ y - 5 = 3(x-4)$ and so $y = 3x - 7$

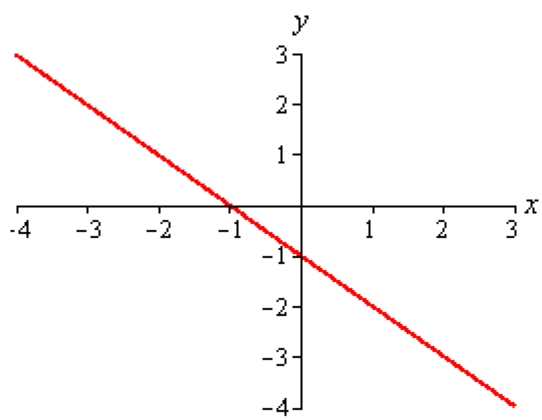


Notes: It doesn't matter which point you use for (x_1, y_1).
If you are given the slope, you only need one point.

Common error: $y - 5 = 3(x-4)$ and so $y = 3x - 17$

Practice: Find the equation of the line joining (−2,1) to (4,−5). Draw the graph.

Answer: $y = -x - 1$

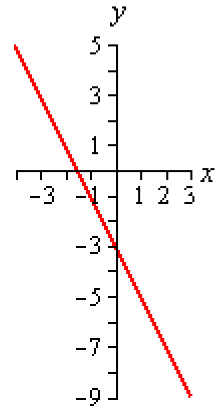


A Great Video for More Detail: https://www.youtube.com/watch?v=4vXqMsvPSv4
A Great Website for More Detail:
https://www.mathwarehouse.com/algebra/linear_equation/write-equation/equation-of-line-given-two-points.php

6) Finding the Equation of a Line Given the Slope and a Point

Problem: Find the equation of the line with slope $m = -2$ passing through the point $(-4,5)$. Draw the graph.

Solution: The easiest formula for finding the equation of a line with slope m and point (x_1, y_1) is $y - y_1 = m(x - x_1)$.



$m = -2$ and $(x_1, y_1) = (-4,5)$

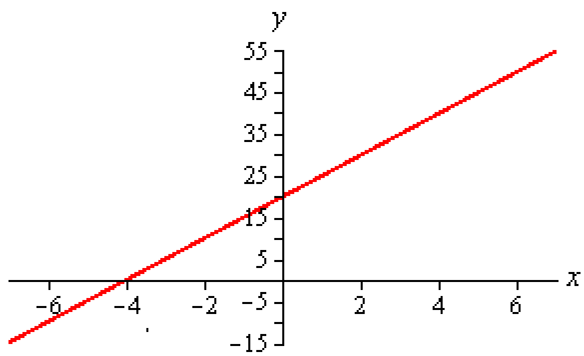
$\therefore\ y - 5 = -2(x + 4)$ and so $y = -2x - 3$.

Notes: Sometimes, you are given a "disguised" point. For example, if we were given x intercept $\frac{13}{2}$, we would use the point $\left(\frac{13}{2}, 0\right)$.

Common error: $y - 5 = -2(x - 4)$ and so $y = 3x + 3$

Practice: Find the equation of the line with slope $m = 5$ and with x intercept -4. Draw the graph.

Answer: $y = 5x + 20$

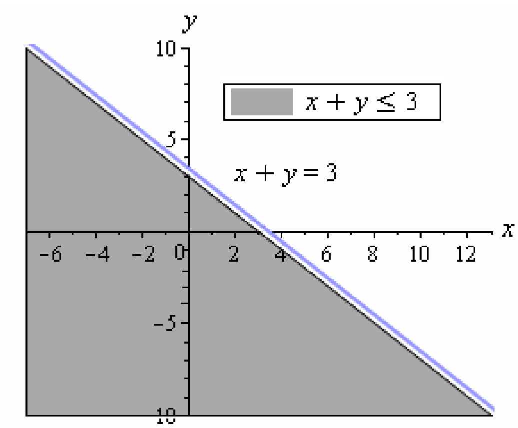


A Great Video for More Detail: https://www.youtube.com/watch?v=H9ym0qevDRE
A Great Website for More Detail: Purplemath.com/modules/strtlneq.htm

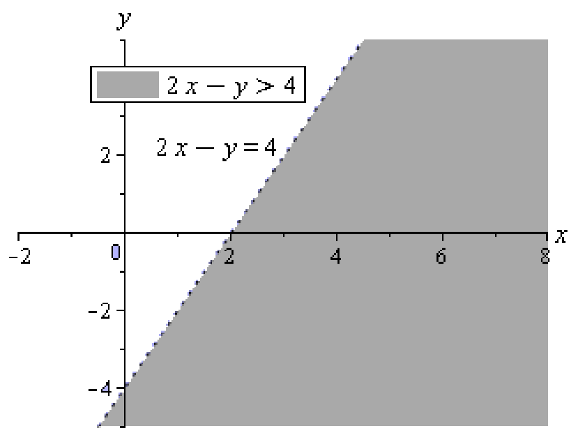
7) Graphing Linear Inequalities

Problem: Show by shading the region in the $x\,y$ plane that satisfies
(i) $x + y \leq 3$ (ii) $2x - y > 4$.

Solution: (i) Draw $x + y = 3$.
Test (0,0) in this equation
Left Side = 0; Right Side =3;
$0 \leq 3$ is **true**. So the shaded region is on the same side of the line as (0,0).



(ii) Draw $2x - y = 4$.
Test (0,0) in this equation.
Left Side = 0; Right Side = 4;
$0 > 4$ is **false**. So the shaded region is on the **opposite** side of the line as (0,0).

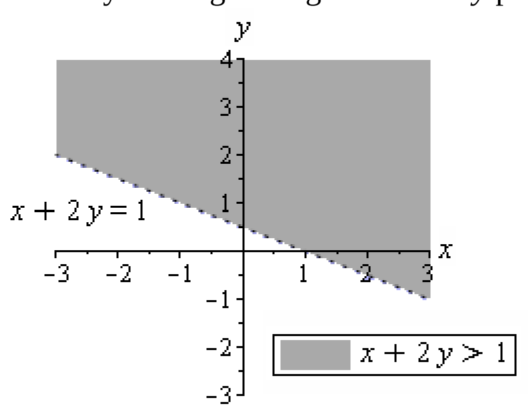


Note: For linear inequalities, draw the line first, dotted if the sign is < or >, solid if the sign is $\leq$ or $\geq$. Then test a point on one side of the line. If the point makes the inequality true, shade that portion of the plane. If not, shade the portion on the other side of the line.

Common error: Testing a point which is on the line. This won't answer the inequality question. You won't get LS < RS or LS > RS; you will get LS = RS, which is no help!

Practice: Show by shading the region in the $x\,y$ plane that satisfies $x + 2y > 1$.

Answer:



A Great Video for More Detail: https://www.youtube.com/watch?v=unSBFwK881s
A Great Website for More Detail: Purplemath.com/modules/syslneq.htm

Solutions Part 3

Algebraic

$$ax^2+bx+c=0 \Leftrightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$|x|<a,\ a>0 \Leftrightarrow -a<x<a$$

$$(a+b)^2=a^2+2ab+b^2$$

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3$$

Skills

$$\begin{array}{r l}
 & x^2-5x+6 \\
x+1 \,\big) & x^3-4x^2+x+6 \\
 & \underline{-(x^3+x^2)} \\
 & \quad -5x^2+x+6 \\
 & \quad \underline{-(-5x^2-5x)} \\
 & \qquad 6x+6 \\
 & \qquad \underline{-(6x+6)} \\
 & \qquad\quad 0
\end{array}$$

$$\therefore\ x^3-4x^2+x+6=(x+1)(x^2-5x+6)$$

1) Adding and Subtracting Like Terms

Problem: Simplify: (i) $4st^2 + 2t^2 s$

(ii) $(x^2 - 3xy + 7x - 1) + (2x^2 - xy - 3y - 4)$

(iii) $3(x+z) + 7(x+z) - y(x+z)$

Solution:
(i) $6st^2$

(ii) $(x^2 - 3xy + 7x - 1) + (2x^2 - xy - 3y - 4) = 3x^2 - 4xy + 7x - 3y - 5$

(iii) $3(x+z) + 7(x+z) - y(x+z) = (10 - y)(x+z)$

Notes: Terms are factors "glued" together with multiplication and division and are separated by addition and subtraction. In (iii), $(x+z)$ is a factor of each term. Don't be confused by the fact that it is composed of two terms inside the brackets.

Common error: In (i), **suppose the question were** $4st^2 + 2ts^2$. In this case, since $4st^2$ and $2ts^2$ are NOT "like" terms, we cannot simplify the expression any further.

Practice: Simplify: $(4w + 3wx - 2) - (2w - 3wx + 1)$

Answer: $2w + 6wx - 3$

A Great Video for More Detail: https://www.youtube.com/watch?v=PkUzlH0iI3E
A Great Website for More Detail: https://www.themathpage.com/Alg/like-terms.htm

2) Multiplying Binomials

Problem: Expand: (i) $(3x+1)(2x-5)$ (ii) $(2a+3b)^2$

Solution:

(i) $(3x+1)(2x-5) = 6x^2 - 15x + 2x - 5 = 6x^2 - 13x - 5$
(ii) $(2a+3b)^2 = (2a+3b)(2a+3b) = 4a^2 + 12ab + 9b^2$

Notes: Each separate term in the first bracket is multiplied with each term in the second. Watch the signs!

Common error: In (ii), $(2a+3b)^2 = 4a^2 + 9b^2$

Practice: Expand: (i) $(a-2b)(a+2b)$ (ii) $(5w-2)^2$

Answer: (i) $a^2 - 4b^2$ ("difference of squares") (ii) $25w^2 - 20w + 4$

A Great Video for More Detail: https://www.youtube.com/watch?v=RTC7RIwdZcE
A Great Website for More Detail: https://www.basic-mathematics.com/multiplying-binomials.html

3) Multiplying Binomials and Trinomials

Problem: Expand: (i) $(x^2+3x+1)(2x-5)$ (ii) $(a+b+c)^2$

Solution: (i) $(x^2+3x+1)(2x-5)=2x^3-5x^2+6x^2-15x+2x-5=2x^3+x^2-13x-5$

(ii) $(a+b+c)^2=(a+b+c)(a+b+c)=a^2+b^2+c^2+2ab+2ac+2bc$

Notes: The number of terms when you expand is the product of the number of terms in each bracket, that is, until you simplify. So there are (3)(2) = 6 terms in (i) and (3)(3) = 9 terms in (ii) before simplification.

Common error: In (ii), $(a+b+c)^2=a^2+b^2+c^2$

Practice: Expand: (i) $(a+2b-c)(a+2b+c)$ (ii) $(a-b-c)^2$

Answer: (i) $a^2+4ab+4b^2-c^2$ (ii) $a^2+b^2+c^2-2ab-2ac+2bc$

A Great Video for More Detail: https://www.youtube.com/watch?v=fGThIRpWEE4
A Great Website for More Detail:
https://courses.lumenlearning.com/prealgebra/chapter/multiplying-a-trinomial-by-a-binomial/

4) Expanding $(a \pm b)^3$

Problem: Expand: (i) $(a+b)^3$ (ii) $(a-b)^3$

Solution:

(i) $(a+b)^3 = (a+b)^2(a+b) = (a^2+2ab+b^2)(a+b)$
$= a^3 + a^2b + 2a^2b + 2ab^2 + b^2a + b^3 = a^3 + 3a^2b + 3ab^2 + b^3$

(ii) $(a-b)^3 = (a-b)^2(a-b) = (a^2-2ab+b^2)(a-b)$
$= a^3 - a^2b - 2a^2b + 2ab^2 + b^2a - b^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Notes: ab^2 and b^2a are "like" terms which is why $2ab^2 + b^2a = 3a^2b$.

Common error: In (i) $(a+b)^3 = a^3 + b^3$

Practice: Expand $(2x-3y)^3$

Answer: $8x^3 - 36x^2y + 54xy^2 - 27y^3$

A Great Video for More Detail: https://www.youtube.com/watch?v=9Ez1Hs5HhyQ
A Great Website for More Detail: https://brilliant.org/wiki/applying-the-perfect-cube-identity/

5) Factoring Simple Trinomials

Problem: Factor: (i) x^2+5x+4 (ii) x^2+3x-4 (iii) $6x^2+17x+5$ (iv) $6x^2-13x-5$

Solution:

(i) $x^2+5x+4=(x+4)(x+1)$

(ii) $x^2+3x-4=(x+4)(x-1)$

(iii) $6x^2+17x+5=(3x+1)(2x+5)$

(iv) $6x^2-13x-5=(3x+1)(2x-5)$

Notes: $(ax+b)(cx+d)=acx^2+(ad+bc)x+bd$
We need to find a and c for the x^2 coefficient and b and d for the constant so that $ad+bc$ is the x coefficient. We do this largely by trial and error. When stuck, there is a sure method. See the next question.

Common error: In (ii) $x^2+3x-4=(x-4)(x+1)$

Practice: Factor: (i) $x^2-2x-15$ (ii) $9x^2+12x+4$

Answer: (i) $(x-5)(x+3)$ (ii) $(3x+2)^2$

A Great Video for More Detail: https://www.youtube.com/watch?v=YtN9_tCaRQc
A Great Website for More Detail: **thematpage.com/alg/factoring-trinomials.htm**

6) Factoring Less Simple Trinomials Using the Quadratic Formula

Problem: Factor: (i) x^2+3x+1 (ii) $6x^2-5x-2$

Hint: **EVERY** quadratic expression of the form ax^2+bx+c can be factored as $a(x-r_1)(x-r_2)$, where $r_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

Solution:

(i) If we solve $x^2+3x+1=0$ using the quadratic formula with $a=1,\ b=3,$ and $c=1,$ we find roots

$$r_1=\frac{-3+\sqrt{9-4(1)(1)}}{2}=\frac{-3+\sqrt{5}}{2} \text{ and } r_2=\frac{-3-\sqrt{9-4(1)(1)}}{2}=\frac{-3-\sqrt{5}}{2}$$

$$\therefore\ x^2+3x+1=(x-r_1)(x-r_2)=\left(x-\frac{-3+\sqrt{5}}{2}\right)\left(x-\frac{-3-\sqrt{5}}{2}\right)=\left(x-\frac{-3+\sqrt{5}}{2}\right)\left(x+\frac{3+\sqrt{5}}{2}\right)$$

(ii) If we solve $6x^2-5x-2=0$ using the quadratic formula with $a=6,\ b=-5,$ and $c=-2,$ we find roots

$$r_1=\frac{5+\sqrt{25-4(6)(-2)}}{12}=\frac{5+\sqrt{73}}{12} \text{ and } r_2=\frac{5-\sqrt{25-4(6)(-2)}}{12}=\frac{5-\sqrt{73}}{12}$$

$$\therefore\ 6x^2-5x-2=6(x-r_1)(x-r_2)=6\left(x-\frac{5+\sqrt{73}}{12}\right)\left(x-\frac{5-\sqrt{73}}{12}\right)$$

Notes: Note in (ii) that we put the original x^2 coefficient, 6, "out front" in the answer. We really reduced the problem to factoring $6\left(x^2-\frac{5}{6}x-\frac{1}{3}\right)$ but working with integers for a, b and c is easier than working with fractions.

Common error: In (ii) $6x^2-5x-2=\left(x-\frac{5+\sqrt{73}}{12}\right)\left(x-\frac{5-\sqrt{73}}{12}\right)$, that is, forgetting to multiply by the x^2 coefficient 6.

Practice: Factor: (i) $x^2-2x-15$ (ii) $3x^2+9x+2$

Answer:

(i) $(x-5)(x+3)$ (ii) $3\left(x-\frac{-9+\sqrt{57}}{6}\right)\left(x-\frac{-9-\sqrt{57}}{6}\right)=3\left(x-\frac{-9+\sqrt{57}}{6}\right)\left(x+\frac{9+\sqrt{57}}{6}\right)$

A Great Video for More Detail: https://www.youtube.com/watch?v=a1nnUA_DYm0
A Great Website for More Detail: **purplemath.com/modules/quadform.htm**

7) Factoring Difference of Squares: $a^2 - b^2 = (a-b)(a+b)$

Problem: 1) Factor: (i) x^2-9 (ii) $x^4-(y+1)^2$

2) Rationalize the denominator: $\dfrac{1}{\sqrt{x}-4}$

Solution:

1)(i) $x^2-9=(x-3)(x+3)$

(ii) $x^4-(y+1)^2 \overset{\boxed{\begin{array}{l} a = x^2 \\ b = y+1 \end{array}}}{=} (x^2-(y+1))(x^2+(y+1))=(x^2-y-1)(x^2+y+1)$

2) $\dfrac{1}{\sqrt{x}-4} \overset{\boxed{\begin{array}{c}\text{Use difference of squares in reverse!} \\ a=\sqrt{x} \quad b=4\end{array}}}{=} \dfrac{\sqrt{x}+4}{\left(\sqrt{x}-4\right)\left(\sqrt{x}+4\right)}=\dfrac{\sqrt{x}+4}{x-16}$

Note: The order of the factors doesn't matter: $a^2-b^2=(a-b)(a+b)=(a+b)(a-b)$

Common error: $x^2-9=(x-3)^2$

Practice: (i) Factor: $25x^2-16y^2$ (ii) Rationalize the denominator: $\dfrac{2}{\sqrt{a}+b}$

Answer: (i) $(5x-4y)(5x+4y)$ (ii) $\dfrac{2(\sqrt{a}-b)}{a-b^2}$

A Great Video for More Detail: https://www.youtube.com/watch?v=_qyVzH3e1dY
A Great Website for More Detail: **purplemath.com/modules/specfact.htm**

8) Factoring Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Problem: (i) Factor: $8x^3 - 27$

(ii) Rationalize the denominator: $\dfrac{1}{\sqrt[3]{x} - 2}$

Solution:

(i) $8x^3 - 27 \overset{\boxed{a=2x,\ b=3}}{=} (2x-3)(4x^2+6x+9)$

(ii) $$\frac{1}{x^{\frac{1}{3}} - 2} \overset{\boxed{\text{Use difference of cubes in reverse! } a = x^{\frac{1}{3}} \text{ and } b = 2 \text{ so that } a^3 = x \text{ and } b^3 = 8}}{=} \frac{\left(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4\right)}{\left(x^{\frac{1}{3}} - 2\right)\left(x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4\right)} = \frac{x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 4}{x - 8}$$

Note: The order of the factors doesn't matter:

$$a^3 - b^3 = (a-b)(a^2+ab+b^2) = (a^2+ab+b^2)(a-b)$$

Common error: $8x^3 - 27 = (2x-3)(4x^2 - 6x + 9)$

Practice: (i) Factor: $x^6 - 64y^3$ (ii) Rationalize the denominator: $\dfrac{2}{a^{\frac{1}{3}} - b^{\frac{1}{3}}}$

Answer: (i) $x^6 - 64y^3 = (x^2 - 4y)(x^4 + 4x^2y + 16y^2)$

(ii) $$\frac{2}{a^{\frac{1}{3}} - b^{\frac{1}{3}}} = \frac{2(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})}{a - b}$$

A Great Video for More Detail: https://www.youtube.com/watch?v=CxcP4ylUP5w
A Great Website for More Detail: **purplemath.com/modules/specfact2.htm**

9) Factoring Sum of Cubes: $\boldsymbol{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$

Problem: (i) Factor: $8x^3 + 27$

(ii) Rationalize the denominator: $\dfrac{1}{\sqrt[3]{x}+2}$

Solution:

(i) $8x^3 + 27 \overset{\boxed{\substack{a=2x\\ b=3}}}{=} (2x+3)(4x^2 - 6x + 9)$

(ii) $$\frac{1}{x^{\frac{1}{3}}+2} \overset{\boxed{\substack{\text{Use sum of cubes in reverse!}\\ a = x^{\frac{1}{3}} \text{ and } b = 2 \text{ so that } a^3 = x \text{ and } b^3 = 8}}}{=} \frac{\left(x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 4\right)}{\left(x^{\frac{1}{3}}+2\right)\left(x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 4\right)} = \frac{x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 4}{x+8}$$

Note: The order of the factors doesn't matter:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) = (a^2 - ab + b^2)(a+b)$$

Common error: $8x^3 + 27 = (2x+3)(4x^2 + 6x + 9)$

Practice: (i) Factor: $x^6 + 64y^3$ (ii) Rationalize the denominator : $\dfrac{2}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}$

Answer: (i) $x^6 + 64y^3 = (x^2 + 4y)(x^4 - 4x^2y + 16y^2)$

(ii) $$\frac{2}{a^{\frac{1}{3}} + b^{\frac{1}{3}}} = \frac{2(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})}{a+b}$$

A Great Video for More Detail: https://www.youtube.com/watch?v=ADj8sGSjewg
A Great Website for More Detail: **purplemath.com/modules/specfact2.htm**

10) Factoring $\boldsymbol{a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \ldots + ab^{n-2} + b^{n-1})}$

Problem: Factor: $x^5 - y^5$

Solution: $x^5 - y^5 = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$

Note: This works for any positive integer *n*. Note the terms on the right are ALL positive.
The exponents on a start at $n-1$ and decrease to 0 and the exponents on b start at 0 and increase to $n-1$.

Common error: $x^5 - y^5 = (x-y)(x^4 + y^4)$

Practice: Factor: $16x^4 - y^4$

Answer: $16x^4 - y^4 = (2x-y)(8x^3 + 4x^2y + 2xy^2 + y^3)$

A Great Video for More Detail: https://www.youtube.com/watch?v=HKDdzBuffoY
A Great Website for More Detail: **mathforum.org/library/drmath/view/55601.html**

11) Factoring $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3 + ... - ab^{n-2} + b^{n-1})$ and n MUST BE ODD!

Problem: Factor: $x^5 + y^5$

Solution: $x^5 + y^5 = (x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

Note: Expressions such as $x^2 + y^2$ and $x^4 + y^4$ (the exponent is even) DO NOT have a factor of $(x+y)$. This method only works when n is an ODD positive integer.

Common error: $x^5 + y^5 = (x+y)(x^4 + y^4)$

Practice: Factor: (i) $32x^5 + y^5$ where one factor is $(2x+y)$. (ii) $16x^4 + y^4$

Answer: (i) $32x^5 + y^5 = (2x+y)(16x^4 - 8x^3y + 4x^2y^2 - 2xy^3 + y^4)$
(ii) You cannot because the exponent is EVEN.

A Great Video for More Detail: https://www.youtube.com/watch?v=sql2LK1Ha6M
A Great Website for More Detail:
https://artofproblemsolving.com/wiki/index.php/Sum_and_difference_of_powers

12) The Factor Theorem: Part 1

Problem: Factor the expression x^3-4x^2+x+6.

Solution: To find a factor, substitute $x=1,\ -1,\ 2,\ -2,\ 3,\ -3,\ 6,\ -6,$ in other words, divisors of 6, into the expression. The Factor Theorem tells us that if $x=a$ makes the expression equal to 0, then $x-a$ is a factor, that is, $x-a$ will divide evenly into x^3-4x^2+x+6. Then we can find the remaining factors without using the Factor Theorem again, because the quotient will be a quadratic.

$x=1$: $\quad 1-4+1+6=4\neq 0$

$x=-1$: $\quad -1-4-1+6=0=0$

Therefore $x+1$ is a factor. Now we divide $x+1$ into x^3-4x^2+x+6 as the first step to finding the other factors.

$$\begin{array}{r} x^2-5x+6 \\ x+1\overline{)\,x^3-4x^2+x+6} \\ \underline{-(x^3+x^2)} \\ -5x^2+x+6 \\ \underline{-(-5x^2-5x)} \\ 6x+6 \\ \underline{-(6x+6)} \\ 0 \end{array}$$

$$\therefore\ x^3-4x^2+x+6=(x+1)(x^2-5x+6)\overset{\boxed{\text{The quadratic is easy to factor!}}}{=}(x+1)(x-2)(x-3)$$

$$\begin{array}{r} 9 \\ 3\overline{)\,27} \\ \underline{-(27)} \\ 0 \end{array}$$

Note: Polynomial division is just like ordinary division. For example,

Common error: Not properly subtracting to get the remainder at each step in the polynomial division.

Practice: Use the Factor Theorem to factor $x^3-7x^2+16x-12$.

Answer: $x^3-7x^2+16x-12=(x-2)^2(x-3)$

A Great Video for More Detail: https://www.youtube.com/watch?v=H9MHo5UM7oI
A Great Website for More Detail: **purplemath.com/modules/factrthm.htm**

13) The Factor Theorem: Part 2

Problem: Find all the **rational** roots of $2x^3-5x^2-4x+3$.

Solution: By the Factor Theorem, all rational roots must be of the form $\frac{a}{b}$, where a is a divisor of 3 and b is a divisor of 2. So we will substitute $x=1,\ -1,\ 3,\ -3,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{3}{2},\ -\frac{3}{2}$ in the original expression. If $x=r$ makes the expression equal to 0, then $x-r$ is a factor and r is a root. Since we have cubic, there are at most three rational factors.

$x=1$: $\quad 2-5-4+3=-4\neq 0$

$x=-1$: $\quad -2-5+4+3=0$

$x=3$: $\quad 54-45-12+3=0$

$x=-3$: $\quad -54-45+12+3\neq 0$

$x=\frac{1}{2}$: $\quad \frac{1}{4}-\frac{5}{4}-2+3=0$

Since we have three roots and the original expression is a cubic polynomial, we are finished.

The rational roots (and in fact all the roots) are $-1,\ 3,$ and $\frac{1}{2}$.

Note: In factored form, $2x^3-5x^2-4x+3=2(x+1)(x-3)\left(x-\frac{1}{2}\right)=(x+1)(x-3)(2x-1)$.

Common error: Here, it is very easy to make arithmetic errors when substituting numbers like $\frac{1}{2}$ into the expression.

Practice: Use the Factor Theorem to find the rational roots of $2x^3-3x^2-x-2$.

Answer: $x=2$; the other two roots are complex numbers.

$2x^3-3x^2-x-2=(2x^2+x+1)(x-2)$

A Great Video for More Detail: https://www.youtube.com/watch?v=icETTMpGXzQ
A Great Website for More Detail: **purplemath.com/modules/factrthm.htm**

14) Polynomial Division

Problem: Divide x^3-5x^2+x+6 by $x-4$. Express your answer $\dfrac{x^3-5x^2+x+6}{x-4}$ both in the form $\dfrac{x^3-5x^2+x+6}{x-4}$ = Quotient + $\dfrac{\text{Remainder}}{\text{Divisor}}$

and $\quad x^3-5x^2+x+6=$ Quotient $\times$ Divisor + Remainder.

Solution:

$$
\begin{array}{r}
x^2-x-3 \\
x-4\overline{\big)\,x^3-5x^2+x+6} \\
-(x^3-4x^2) \\
\hline
-x^2+x+6 \\
-(-x^2+4x) \\
\hline
-3x+6 \\
-(-3x+12) \\
\hline
-6
\end{array}
$$

Therefore, $\dfrac{x^3-5x^2+x+6}{x-4}=x^2-x-3-\dfrac{6}{x-4}$

and $x^3-5x^2+x+6=(x-4)(x^2-x-3)-6$

Note: Compare $\dfrac{28}{3}$ = Quotient + $\dfrac{\text{Remainder}}{\text{Divisor}}=9+\dfrac{1}{3}$ with

$\dfrac{x^3-5x^2+x+6}{x-4}$ = Quotient + $\dfrac{\text{Remainder}}{\text{Divisor}}=x^2-x-3-\dfrac{6}{x-4}$.

Common error: Not properly subtracting to get the remainder at each step in the polynomial division.

Practice: Simplify $\dfrac{x^3-2x^2-3x+3}{x+2}$ and express your answer as in the example.

Answer: $\dfrac{x^3-2x^2-3x+3}{x+2}=x^2-4x+5-\dfrac{7}{x+2}$

$x^3-2x^2-3x+3-=\left(x^2-4x+5\right)(x+2)-7$

A Great Video for More Detail: https://www.youtube.com/watch?v=smsKMWf8ZCs
A Great Website for More Detail: https://www.mathsisfun.com/algebra/polynomials-division-long.html

15) Solving for the Roots of a Polynomial

Problem: Solve: (i) $x^2 - 5x + 6 = 0$ (ii) $x^2 - 5x + 1 = 0$ (iii) $x^3 - 7x = -6$

Solution: (i) Factoring, $x^2 - 5x + 6 = (x-3)(x-2) = 0$ and so $x = 2$ or $x = 3$.

(ii) Using the quadratic formula with $a = 1$ and $b = -5$ and $c = 1$, we get

$$x = \boxed{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(1)}}{2(1)} = \frac{5 \pm \sqrt{21}}{2},$$ that is, the roots are

$x = \dfrac{5 + \sqrt{21}}{2}$ and $x = \dfrac{5 - \sqrt{21}}{2}$.

(iii) Apply the Factor Theorem to $P(x) = x^3 - 7x + 6$:

$P(1) = 0$ $P(-1) = 12$ $P(2) = 0$ $P(-2) = 12$ $P(3) = 12$ $P(-3) = 0$

We can stop even though we haven't checked $P(6)$ nor $P(-6)$ (remember that we check all divisors of 6 because we KNOW a cubic polynomial has exactly three roots. So the roots are 1, 2, and -3.

Note 1: $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ has roots $r_1, r_2, r_3, ...,$ and r_n $\Leftrightarrow$

$P(x) = a_n (x - r_1)(x - r_2)(x - r_3)...(x - r_n)$

Note 2: $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ has roots $r_1, r_2, r_3, ...,$ and r_n $\Leftrightarrow$

the x intercepts of $y = P(x)$ are the REAL numbers in the set $\{r_1, r_2, r_3, ..., r_n\}$

Common errors: (i) Getting the signs wrong when factoring a quadratic. (ii) Substituting incorrectly when using the quadratic formula. (iii) Not evaluating $P(a)$ correctly when using the Factor Theorem.

Practice: Solve: (i) $x^2 + 11x + 30 = 0$ (ii) $x^2 - 2x + 2 = 0$ (iii) $x^3 - x^2 + x - 1 = 0$

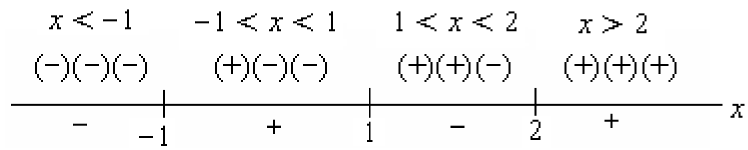
Answer: (i) $-5, -6$ (ii) $1+i, 1-i$ (iii) $1, i, -i$ (Remember, $i = \sqrt{-1}$.)

A Great Video for More Detail: https://www.youtube.com/watch?v=x9lb_frpkH0
A Great Website for More Detail: https://www.mathsisfun.com/algebra/polynomials-solving.html

16) Solving "Factored" Inequalities: Numerators Only

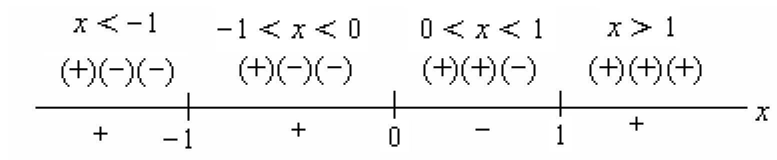
Problem: Solve (i) $(x+1)(x-1)(x-2) \geq 0$ (ii) $(x+1)^2 x(x-1)^3 > 0$

Solution: (i) The only x values where the expression can change from + to − or − to + are −1, 1, and 2. So we analyze the sign of each factor in the expression on a number line using these values.



The solution in interval notation is $[-1,1] \cup [2,\infty)$.

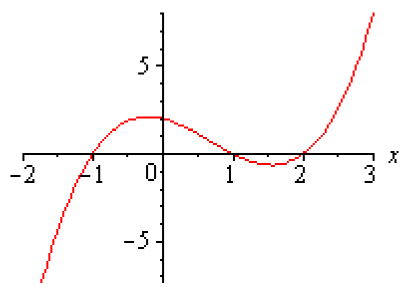
(ii) The only x values where the expression can change from + to − or − to + are −1, 0, and 1. **However,** the exponent on x + 1 is even: $(x+1)^2 \geq 0$ always!



The solution in interval notation is $(-\infty,-1) \cup (-1,0) \cup (1,\infty)$.

Note 1: If one of the factors is, for example, $(x+1)^{1/3}$, then the term will change sign at $x = -1$. If the factor were $(x+1)^{2/3}$, it would not change sign.

Note 2: What is the geometric significance of this kind of problem? In (i), you have found where the graph of $y = (x+1)(x-1)(x-2)$ is on or above the x axis.



Common error: Getting the signs wrong when the exponent on a factor is even.

Practice: (i) $-(x-2)^3 x(x-3)^{1/5} \geq 0$ (ii) $-(x-2)^4 x(x-3)^{2/5} \geq 0$

Answer: (i) $(-\infty,0] \cup [2,3]$ (ii) $(-\infty,0] \cup \{2,3\}$

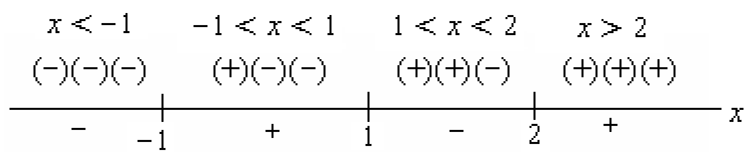
Note: Because the expression in (ii) equals 0 at $x = 2$ and $x = 3$ (as well as $x = 0$), 2 and 3 are part of the solution.

A Great Video for More Detail: https://www.youtube.com/watch?v=U9UtNsXLg5E
A Great Website for More Detail: **purplemath.com/modules/ineqsolv.htm**

17) Solving "Factored" Rational Inequalities:

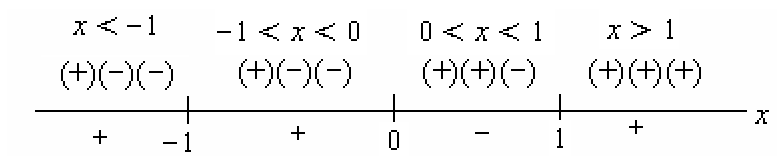
Problem: Solve: (i) $\dfrac{(x+1)(x-2)}{(x-1)} \geq 0$ (ii) $\dfrac{(x+1)^2(x-1)^3}{x} > 0$

Solution: (i) The only x values where the expression can change from + to − or − to + are −1, 1, and 2. So we analyze the sign of the each factor in the expression on a number line using these values.



The question asks for values of x where the expression is "≥ 0". So values of x where the numerator is 0 work. Values of x where the denominator is 0 **do not!** We include $x = -1$ and $x = 2$ but not $x = 1$. The solution in interval notation is $[-1,1) \cup [2,\infty)$.

(ii) The only x values where the expression can change from + to − or − to + are −1, 0, and 1. **However,** the exponent on $x + 1$ is even: $(x+1)^2 \geq 0$ always!



The question asks for values of x where the expression is "> 0".So values of x where the numerator is 0 **do not work.** Values of x where the denominator is 0 **never work!** We exclude $x = -1, 0, 1$ The solution in interval notation is $(-\infty,-1) \cup (-1,0) \cup (1,\infty)$.

Note 1: If one of the factors is, for example, $(x+1)^{1/4}$, then $x \geq -1$ because you can't take an even root of a negative number.

Note 2: If one of the factors is, for example, $(x+1)^{1/3}$, then the term will change sign at $x = -1$. If the factor were $(x+1)^{2/3}$, it would not change sign.

Common error: Getting the signs wrong when the exponent on a factor is even.

Practice: (i) $\dfrac{-(x-3)^{1/5}}{x(x-2)^3} \geq 0$ (ii) $\dfrac{-(x-3)^{2/5}}{(x-2)^4 x} \geq 0$

Answer: (i) $(-\infty,0) \cup (2,3]$ (ii) $(-\infty,0) \cup \{3\}$

A Great Video for More Detail https://www.youtube.com/watch?v=ZjeMdXV0QMg
A Great Website for More Detail: **purplemath.com/modules/ineqrtnl.htm**

18) Completing the Square

Problem: Complete the square: (i) $x^2 - 8x + 25$ (ii) $3x^2 + 36x - 17$ (iii) $-2x^2 + 3x + 1$

Solution: Remember $(x+a)^2 = x^2 + 2ax + a^2$

$$\text{(i) } x^2 - 8x + 25 \overset{\boxed{\begin{array}{l}2a=-8\\ \therefore a=-4\\ \text{and } a^2=16\end{array}}}{=} x^2 \overset{\boxed{2a}}{-8x} + \overset{\boxed{a^2}}{16} + 25 \quad \overset{\boxed{\begin{array}{l}\text{We added 16 so we}\\ \text{subtract 16 to compensate!}\end{array}}}{-16} \quad = (x \overset{\boxed{a}}{-4})^2 + 9$$

$$\text{(ii) } 3x^2 + 36x - 17 = 3(x^2 \overset{\boxed{\begin{array}{l}2a=12\\ \therefore a=6\\ \text{and } a^2=36\end{array}}}{} + 12x) - 17$$

$$= 3(x^2 + 12x + 36) - 17 \quad \overset{\boxed{\begin{array}{l}\text{We added } 3\times 36=108 \text{ so}\\ \text{we subtract 108 to compensate.}\end{array}}}{-108} \quad = 3(x+6)^2 - 125$$

$$\text{(iii) } -2x^2 + 3x + 1 = -2(x^2 \quad \overset{\boxed{\begin{array}{l}2a=-\frac{3}{2}\ \therefore a=-\frac{3}{4}\\ \text{and } a^2=\frac{9}{16}\end{array}}}{-\frac{3}{2}} \quad x\ ) + 1$$

$$= -2(x^2 - \frac{3}{2}x + \frac{9}{16}) + 1 \quad \overset{\boxed{\begin{array}{l}\text{Because of the } -2 \text{ outside the bracket, we}\\ \text{subtracted } 2\left(\frac{9}{16}\right) = \frac{9}{8} \text{ so we add } \frac{9}{8} \text{ to compensate.}\end{array}}}{+\frac{9}{8}} \quad = -2\left(x - \frac{3}{4}\right)^2 + \frac{17}{8}$$

Note: When completing the square on the expression $Ax^2 + Bx + C$, start this way: $Ax^2 + Bx + C = A\left(x^2 + \frac{B}{A}x\right) + C$, that is, factor the A from the x^2 and x terms, but not the constant.

Common error: Not "compensating" correctly

Practice: (i) $x^2 - 4x - 3$ (ii) $-3x^2 + 5x + 1$

Answer: (i) $(x-2)^2 - 7$ (ii) $-3\left(x - \frac{5}{6}\right)^2 + \frac{37}{12}$

A Great Video for More Detail: https://www.youtube.com/watch?v=prx_Bf2hakw
A Great Website for More Detail: **purplemath.com/modules/sqrquad.htm**

19) Adding and Subtracting Rational Expressions

Problem: By getting a common denominator, simplify the following expressions:

(i) $\frac{1}{3x}-\frac{1}{2y}+\frac{1}{6z}$ (ii) $\frac{2x+1}{x-1}-\frac{x+1}{x+2}-\frac{5x+4}{x^2+x-2}$

Solution: (i) $\frac{1}{3x}-\frac{1}{2y}+\frac{1}{6z}$

$$\overset{\boxed{\text{Get the lowest common denominator!}}}{=}\ \frac{2yz}{6xyz}-\frac{3xz}{6xyz}+\frac{xy}{6xyz}\ \overset{\boxed{\text{Simplify the numerator.}}}{=}\ \frac{2yz-3xz+xy}{6xyz}$$

(ii) $\frac{2x+1}{x-1}-\frac{x+1}{x+2}-\frac{5x+4}{x^2+x-2}$

$$\overset{\boxed{\text{Factor the denominators!}}}{=}\ \frac{2x+1}{x-1}-\frac{x+1}{x+2}-\frac{5x+4}{(x-1)(x+2)}$$

$$\overset{\boxed{\text{Get the lowest common denominator.}}}{=}\ \frac{(2x+1)(x+2)}{(x-1)(x+2)}-\frac{(x+1)(x-1)}{(x-1)(x+2)}-\frac{5x+4}{(x-1)(x+2)}$$

$$\overset{\boxed{\text{Expand the numerator.}}}{=}\ \frac{2x^2+5x+2-(x^2-1)-(5x+4)}{(x-1)(x+2)}$$

$$\overset{\boxed{\text{Now simplify the numerator.}}}{=}\ \frac{x^2-1}{(x-1)(x+2)}$$

$$\overset{\boxed{\text{Check for and divide out any further common factors.}}}{=}\ \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x+2)}=\frac{x+1}{x+2}$$

Note: Adding and subtracting and multiplying and dividing expressions use **exactly** the same methods as adding and subtracting ordinary numerical fractions!

Common error: Not using the lowest common denominator which leads to a more complicated expression and more chances of making a mistake!

Practice: (i) $\frac{3}{2a}-\frac{1}{3b}+\frac{5}{6ab}$ (ii) $\frac{1}{x^2}+\frac{x-2}{x(x+2)}-\frac{x}{(x+2)^2}$

Answer: (i) $\frac{9b-2a+5}{6ab}$ (ii) $\frac{x^2+4}{x^2(x+2)^2}$

A Great Video for More Detail: https://www.youtube.com/watch?v=XTZl7Kn6u4Y
A Great Website for More Detail: **purplemath.com/modules/rtnladd.htm**

20) Multiplying and Dividing Rational Expressions

Problem: Simplify the following rational expressions:

(i) $\dfrac{(x^2-16)}{(x-4)^3}\times\dfrac{(x^2-4x)^2}{x^3+64}$ (ii) $\dfrac{x^2+5xy+4y^2}{x^2+4xy+4y^2}\div\dfrac{x^2+xy}{x^2+2xy}$

Solution: (i) $\dfrac{(x^2-16)}{(x-4)^3}\times\dfrac{(x^2-4x)^2}{x^3+64}$

$$=\frac{\cancel{(x-4)}\,\cancel{(x+4)}}{\cancel{(x-4)^3}}\times\frac{x^2\,\cancel{(x-4)^2}}{\cancel{(x+4)}\,(x^2-4x+16)}=\frac{x^2}{x^2-4x+16}$$

(ii) $\dfrac{x^2+5xy+4y^2}{x^2+4xy+4y^2}\div\dfrac{x^2+xy}{x^2+2xy}$

Factor!
$$=\frac{(x+4y)(x+y)}{(x+2y)^2}\div\frac{x(x+y)}{x(x+2y)}$$

Invert and multiply.
$$=\frac{(x+4y)(x+y)}{(x+2y)^2}\times\frac{x(x+2y)}{x(x+y)}$$

Divide out common factors.
$$=\frac{x+4y}{x+2y}$$

Note: Adding and subtracting and multiplying and dividing expressions use **exactly** the same methods as adding and subtracting ordinary numerical fractions!

Common error: Students often divide out "terms" instead of "factors". For example, in the original question (i), x^2 is a term, while (x^2-16) is a factor.

Practice: Simplify the following rational expressions:

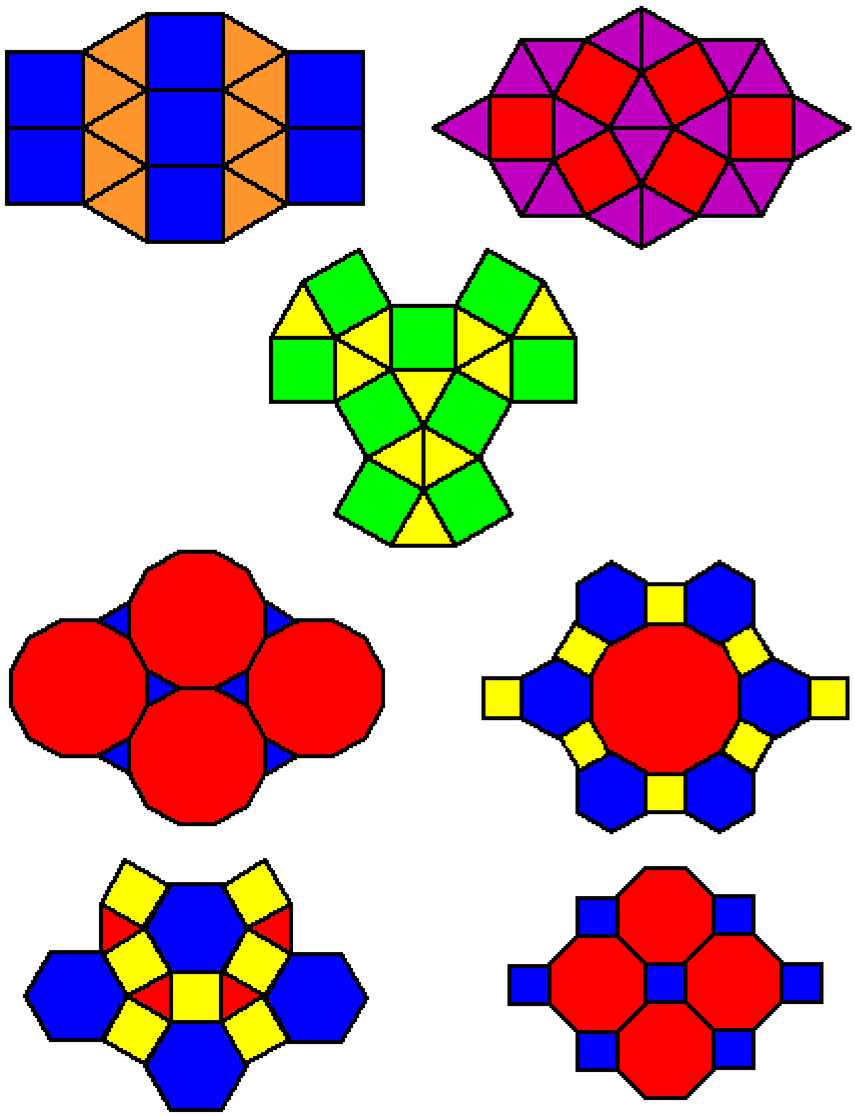
(i) $\dfrac{(x^2+7x+6)^2}{(x^2+12x+36)(x^2-1)}\times\dfrac{x^3-1}{x^2+x+1}$ (ii) $\dfrac{(z^3+5z^2)^3}{z^2+10z+25}\div\dfrac{z^8(z^3+125)}{z^2-5z+25}$

Answer: (i) $x+1$ (ii) $\dfrac{1}{z^2}$

A Great Video for More Detail: https://www.youtube.com/watch?v=WbvLjmK4Kmc
A Great Website for More Detail: **purplemath.com/modules/rtnlmult.htm**

Solutions Part 4

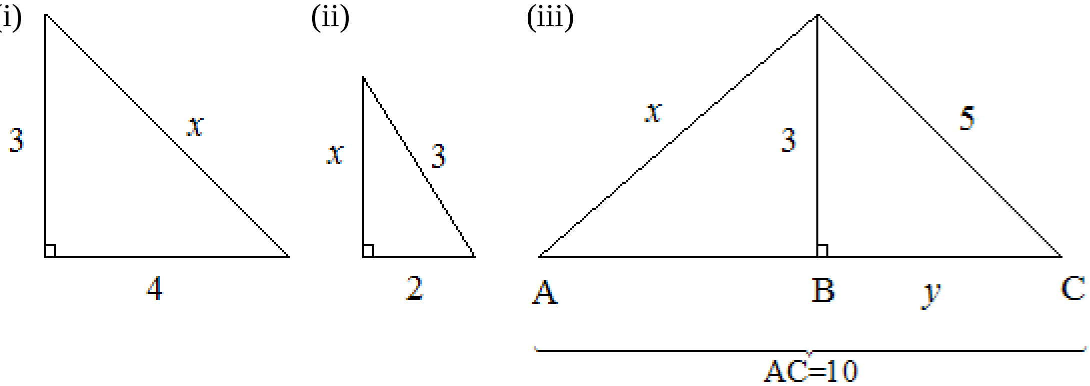
Geometry



1) Pythagorean Theorem

Problem: In the following diagrams find the value of the unknowns:

(i) (ii) (iii)



Solution: (i) $x^2 = 3^2 + 4^2 = 25 \therefore x = 5$

(ii) $3^2 = 2^2 + x^2 \quad \therefore x^2 = 9 - 4 = 5 \therefore x = \sqrt{5}$

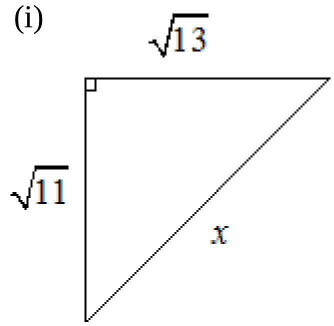
(iii) Let BC = y. Then $5^2 = 25 = y^2 + 3^2$. $\therefore y^2 = 25 - 9 = 16 \therefore y = 4$
AB = $10 - y = 10 - 4 = 6$. Now we can find x: $x^2 = 3^2 + 6^2 = 45 \therefore x = \sqrt{45} = 3\sqrt{5}$.

Note: The important point is that if the length of any two sides of a right angled triangle is known we can determine the third side using the Pythagorean Theorem.

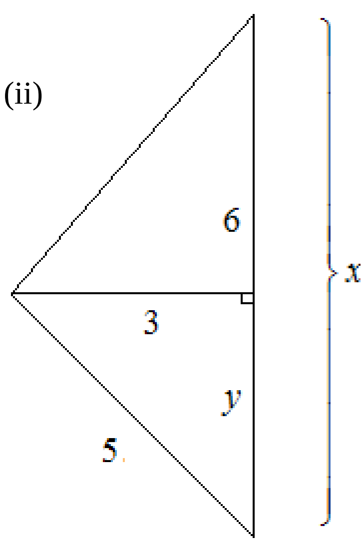
Common error: $x^2 = a^2 + b^2 \therefore x = a + b$

Practice:
Find x in the following:

(i)



(ii)



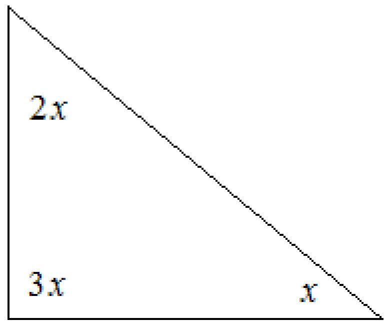
Answer: (i) $x = 2\sqrt{6}$ iii) $x = 10$

A Great Video for More Details: https://www.youtube.com/watch?v=JH9V3bWA1T0
A Great Website for More Details: **purplemath.com/modules/perimetr3.htm**

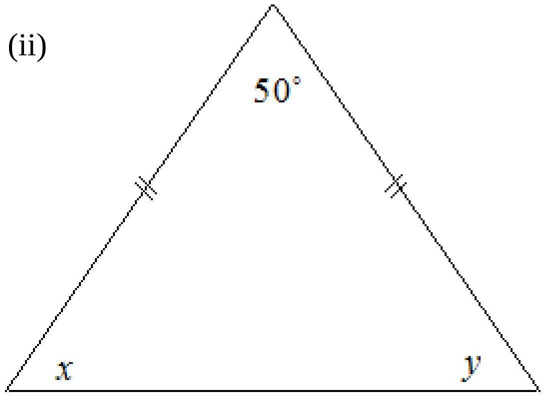
2) Angles in a Triangle

Problem: Find the values of angles x and y (in degree measure) from the following diagrams:

(i)



(ii)



Solution: (i) $x + 2x + 3x = 180°$ $\therefore 6x = 180^\circ$ $\therefore x = 30^\circ$.

(ii) $x + y + 50° = 180°$ But $x = y$ (Isosceles Triangle!)

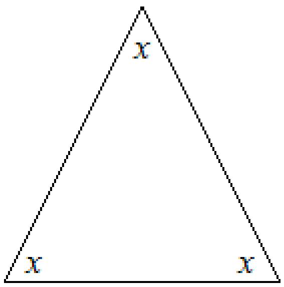
So $2x + 50° = 180°$ and $x = y = 65°$.

Note: Once you know any two angles of a triangle you know all three.

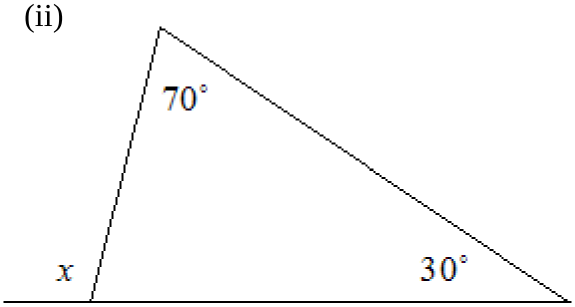
Common error: Making a mistake when solving the equation for the unknown.

Practice: Find x in each of the following diagrams:

(i)



(ii)



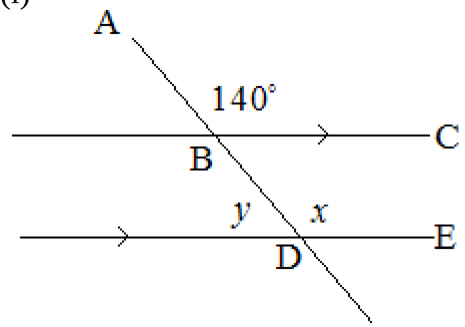
Answer: (i) $x = 60°$ (ii) $x = 100°$

A Great Video for More Details: https://www.youtube.com/watch?v=bp5UxYKPie8
A Great Website for More Details: **mathopenref.com/triangleinternalangles.html**

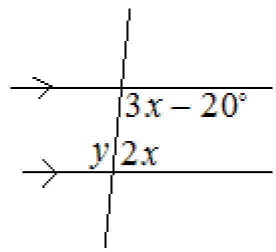
3) The Parallel Line Theorem

Problem: Find the values in degrees of x and y in the following diagrams:

(i)



(ii)



Solution: i) $\angle ABC = \angle BDE = x$ (corresponding angles)

$\therefore x = 140^\circ$ Also $x + y = 180°$ and so $y = 40°$.

(ii) $y = 3x – 20°$ (alternate angles) Also, $2x + y = 180°$.

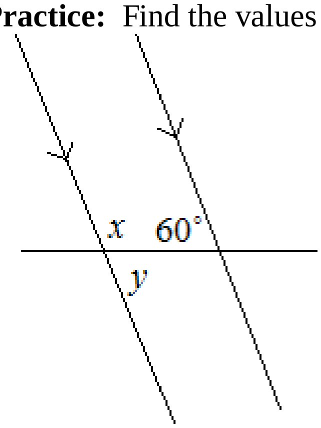
$\therefore 2x + 3x - 20^\circ = 180^\circ \Rightarrow 5x = 200^\circ \Rightarrow x = 40^\circ$

So $x = 40°$ and $y = 3(40°) − 20° = 100°$.

Note: When two lines are parallel, the sum of the interior angles on the same side of the transversal sum to 180°. So we could have solved (ii) by writing $2x + 3x – 20° = 180°$ and then finding first x and then y.

Common error: Misidentification of corresponding and/or alternate angles.

Practice: Find the values in degrees of x and y:

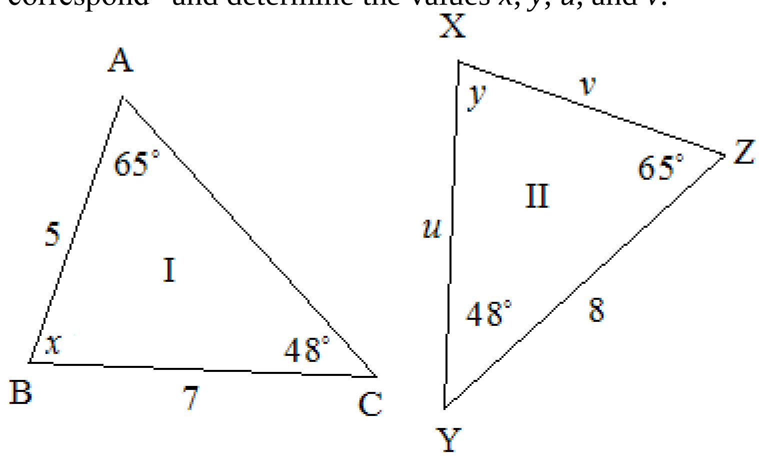


Answer: $x = 120^\circ$, $y = 60^\circ$

A Great Video for More Details: https://www.youtube.com/watch?v=lvg2ycZ2nWA
A Great Website for More Detail:
https://www.mathsisfun.com/geometry/parallel-lines.html

4) Congruent Triangles

Problem: Triangles I and II are congruent. Name the congruent triangles so that the vertices "correspond" and determine the values *x*, *y*, *u*, and *v*.



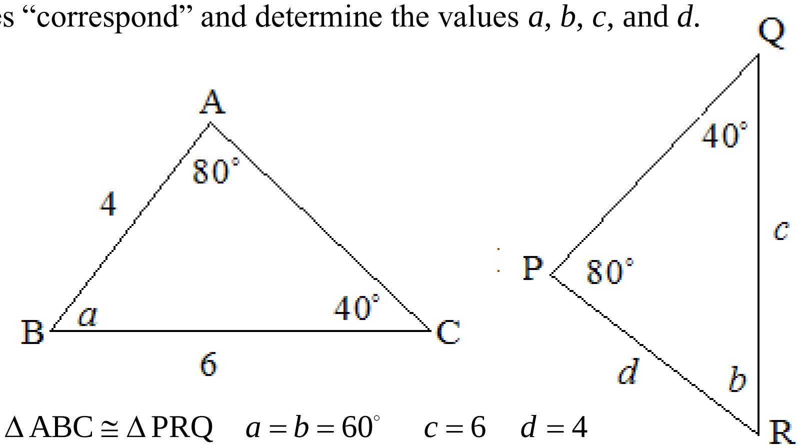
Solution: Note that the two triangles are congruent by AAS: $\Delta\,\text{ABC} \cong \Delta\,\text{ZXY}$

$x = 180^\circ - 48^\circ - 65^\circ = 67^\circ = y \qquad u \overset{\boxed{u=\text{XY}=\text{BC}}}{=} 7 \qquad v \overset{\boxed{u=\text{XZ}=\text{BA}}}{=} 5$

Notes: When two congruent triangles are superimposed, every corresponding component matches up: sides, angles, area, perimeter, **everything.** Congruency can be determined by SSS, SAS, ASA, and AAS but **not by SSA nor by AAA!**

Common error: $\Delta\,\text{ABC} \cong \Delta\,\text{ZYX}$. This is wrong because the vertices do not correspond.

Practice: The two triangles below are congruent. Name the congruent triangles so that the vertices "correspond" and determine the values *a*, *b*, *c*, and *d*.

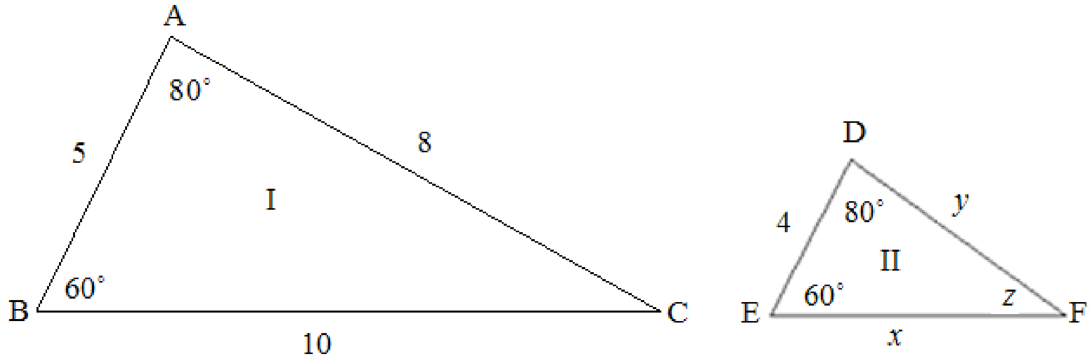


Answer: $\Delta\,\text{ABC} \cong \Delta\,\text{PRQ} \quad a = b = 60^\circ \quad c = 6 \quad d = 4$

A Great Video for More Details: https://www.youtube.com/watch?v=vGuiy7NnJlM
A Great Website for More Detail:
https://www.mathsisfun.com/geometry/triangles-congruent-finding.html

5) Similar Triangles

Problem: Triangles I and II are similar. Name the similar triangles so that the vertices "correspond" and determine the values *x*, *y*, and *z*.



Solution: Note that the two triangles are similar by AA: $\Delta\,\mathrm{ABC} \sim \Delta\,\mathrm{DEF}$

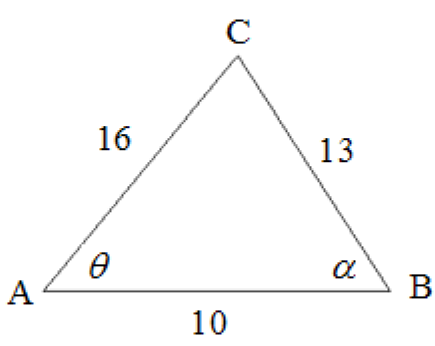
$z = 180^\circ - 80^\circ - 60^\circ = 40^\circ$ By similar triangles, $\dfrac{x}{10} = \dfrac{y}{8} = \dfrac{4}{5}$

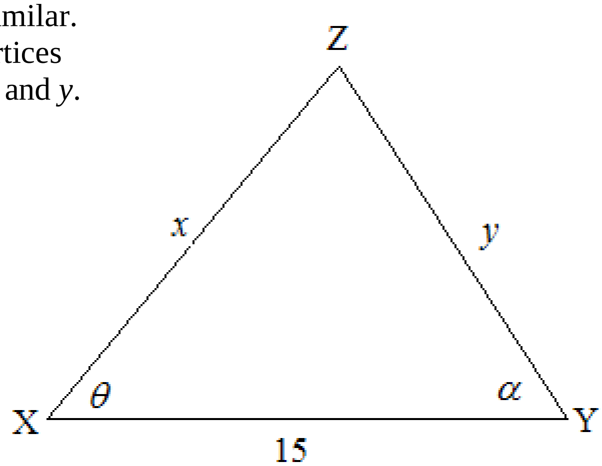
Solving gives $x = 8$ and $y = \dfrac{32}{5}$.

Note: Similarity can be determined by SSS, SAS, and AA (which is the same as AAA!) Here, when we use S, we are referring not to equality of sides but to the ratio of corresponding sides.

Common error: $\Delta\,\mathrm{ABC} \sim \Delta\,\mathrm{EDF}$. This is wrong because the vertices do not correspond.

Practice: The two triangles below are similar. Name the similar triangles so that the vertices "correspond" and determine the values *x* and *y*.





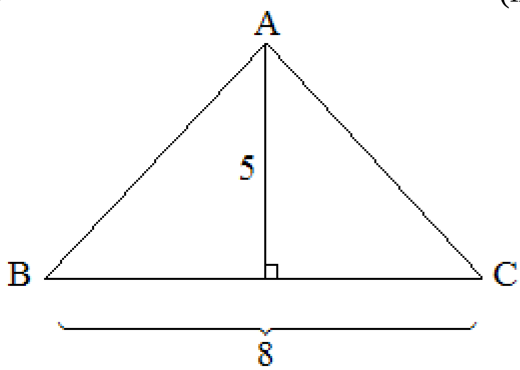
Answer: $\Delta\,\mathrm{ABC} \sim \Delta\,\mathrm{XYZ}$ $x = 24$ $y = \dfrac{39}{2}$

A Great Video for More Details: https://www.youtube.com/watch?v=BI-rtfZVXy0
A Great Website for More Detail:
https://www.mathsisfun.com/geometry/triangles-similar.html

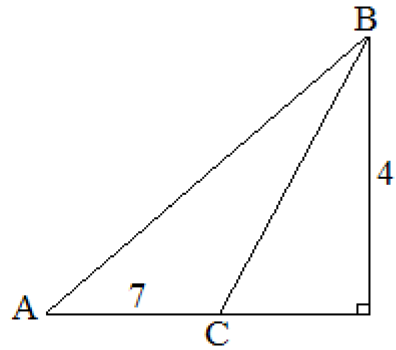
6) Area of a Triangle

Problem: Find the area of ΔABC:

(i)



(ii)

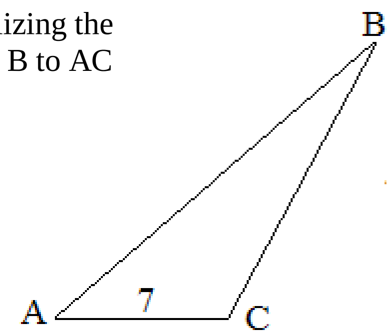


Solution: (i) Area $A = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2} \times 8 \times 5 = 20$ square units.

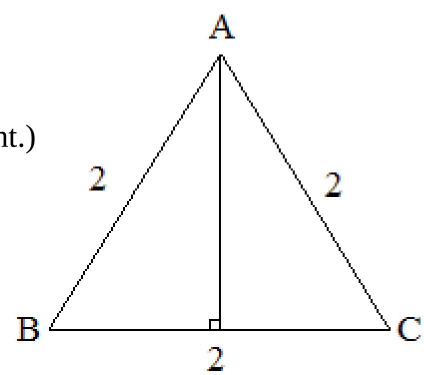
(ii) Area $A = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2} \times 7 \times 4 = 14$ square units.

Note: Choose one side to be the base. Then the height is the **length of the perpendicular** from the vertex opposite to the base. Sometimes the foot of the perpendicular is on an extension of the base as in diagram (ii).

Common error: Not realizing the perpendicular height from B to AC is on an extension of AC.



Practice: Find the area of ΔABC.
(Hint: first use the Pythagorean Theorem to find the perpendicular height.)



Answer: Area = $\sqrt{3}$ sq. units.

A Great Video for More Details: https://www.youtube.com/watch?v=_ywdqdLrKxY
A Great Website for More Detail:
https://www.mathsisfun.com/algebra/trig-area-triangle-without-right-angle.html

7) Area and Circumference of a Circle

Problem: (i) Find the circumference and area of a circle of radius 8 cm.

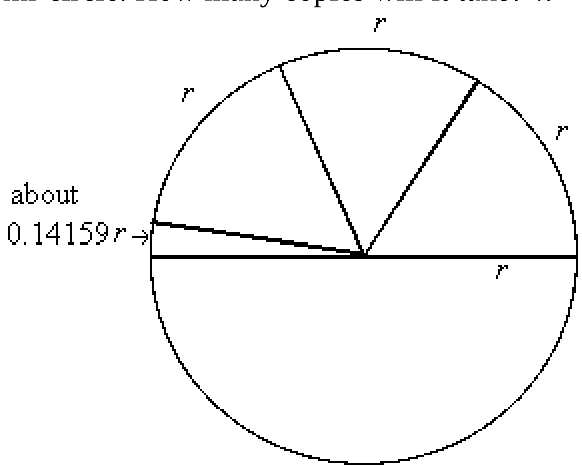
(ii) Find the circumference and area of a circle of diameter 8 m.

(iii) If the area of a circle is 16π, find the radius.

Solution: (i) Radius r = 8 cm
Circumference C $= 2\pi r = 2\pi\times 8 = 16\pi$ cm and area A $= \pi r^2 = \pi\times 8^2 = 64\pi$ cm^2
(ii) Diameter d = 8 Circumference C $= \pi d = \pi\times 8 = 8\pi$ m

Area A $= \pi\times\frac{d^2}{4} = \pi\times\frac{8^2}{4} = 16\pi$ m^2
(iii) Area A = $\pi r^2 = 16\pi$ Then $r^2 = 16$ and r = 4 m.

Note: Take any circle—ANY CIRCLE!—and wrap copies of the radius around the semi-circle. How many copies will it take? $\pi = 3.14159...$



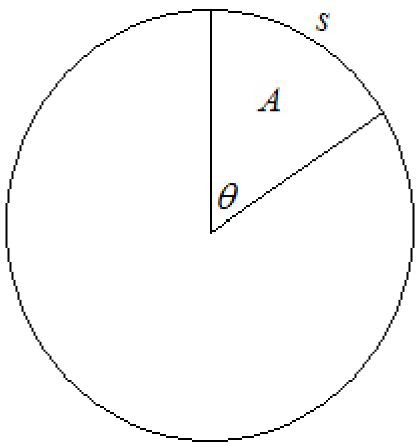
Common error: Using the diameter value d in the formulas $2\pi r$ and πr^2; similarly using the radius in the formulas πd and $\frac{\pi d^2}{4}$.
Practice: Find the circumference and area of a circle of diameter 12 units.

Answer: C $= 12\pi$ units and A= 36π units^2

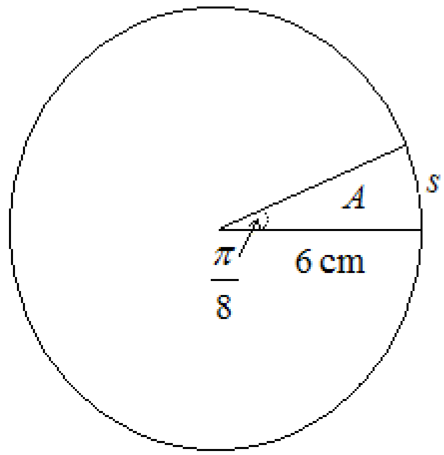
A Great Video for More Details: https://www.youtube.com/watch?v=MQptV71V8Go
A Great Website for More Detail: https://sciencing.com/calculate-area-circumference-circle-7274267.html

8) Arc Length and Area of a Sector of a Circle

Problem: (i) State the arc length *s* and the area of the sector of this circle. Assume θ is in **radian** measure.



(ii) In the circle below, find the arc length *s* and the area *A*.



Solution: (i) Arc length $s = r\theta$ units and $A = \frac{1}{2}r^2\theta$ units^2. (Remember: the angle must be measured in radians.)

(ii) $s = r\theta = 6\times\frac{\pi}{8} = \frac{3\pi}{4}$ cm; $A = \frac{1}{2}r^2\theta = \frac{1}{2}(6)^2\left(\frac{\pi}{8}\right) = \frac{9\pi}{4}\text{cm}^2$

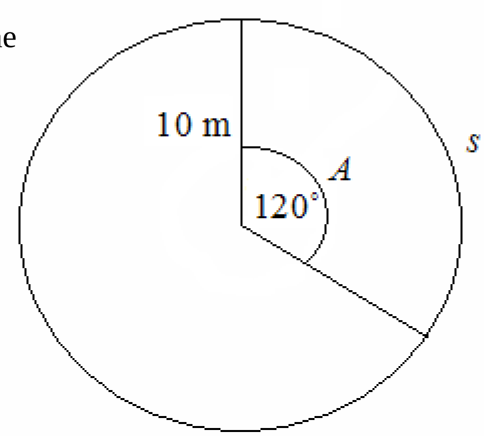
Note: In the correct use of the formulas in (i), θ must be in radians. If θ is given in degrees, convert to radians using this conversion formula:

angle in radians $= \theta\times\frac{\pi}{180^\circ}$.

(VERY) Common error: Using θ in degrees in the formulas in (i).

Practice: In the circle to the right, find the length *s* corresponding sector area *A*.

(Hint: $120^\circ = \frac{2\pi}{3}$ radians)



Answer: $s = \frac{20\pi}{3}$ m ; $A = \frac{100\pi}{3}$ m^2

A Great Video for More Details: https://www.youtube.com/watch?v=00KCeY66FcY
A Great Website for More Details:
https://tutors.com/math-tutors/geometry-help/area-of-a-sector-of-a-circle-formula

9) Volume of a Sphere, Box, Cone, Cylinder

Problem: Find the volume V of
(i) a sphere of radius $r = 3$ cm
(ii) a rectangular box with length $l = 8$ cm, width $w = 5$ cm, and height $h = 50$ cm
(iii) a right circular cone with height $h = 3$ cm and base radius $r = 0.04$ m
(iv) a circular cylinder with height $h = 0.2$ m and radius $r = 4$ cm.

Solution: i) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(3^3\right) = 36\pi \text{ cm}^3$

(ii) $V = l \times w \times h = 8 \times 5 \times 50 = 2000 \text{ cm}^3$

(iii) Get common units! $r = 0.04$ m $= 4$ cm $\therefore V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 4^2 \times 3 = 16\pi \text{ cm}^3$

(iv) $h = 0.2$ m $= 20$ cm $\therefore V = \pi r^2 h = \pi \times 4^2 \times 20 = 320\pi \text{ cm}^3$

Note: You have used a lot of volume formulas (like the cone and the sphere) as early as grade 3. In calculus, you finally get to **PROVE** them.

Common error: Not using a common measurement unit (such as cm) for all the variables in the volume formula.

Practice: Find the volume V of

(i) a sphere of radius 2 cm

(ii) a box of length 0.3 m, width 7 cm, and height 3 cm

(iii) a cone of height 8 cm and radius 4 cm

(iv) a circular cylinder of height 12 cm and base radius 3 cm

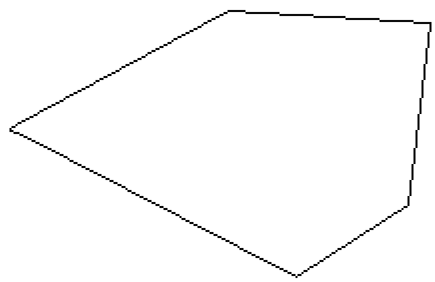
Answers: i) $\frac{32\pi}{3} \text{ cm}^3$ ii) 630 cm^3 iii) $\frac{128\pi}{3} \text{ cm}^3$ iv) $108\pi \text{ cm}^3$

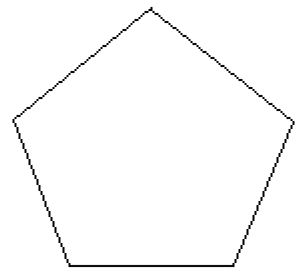
A Great Video for More Details: https://www.youtube.com/watch?v=aPXvrFMyhSU
A Great Website for More Detail: https://www.mathsisfun.com/geometry/cone-sphere-cylinder.html

10) Angles of a Polygon

Problem: i) Find the sum of the interior angles of a pentagon.

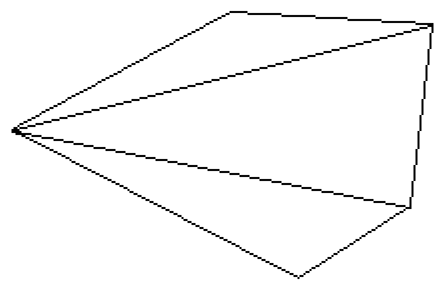
(ii) If all the interior angles of a pentagon are equal, how much is each interior angle?





Solution:

(i) The angles in a triangle sum to 180 degrees. Divide the pentagon into 3 triangles as shown. The sum of the interior angles of the pentagon is equal to $180 \times 3 = 540^{\circ}$



(ii) Since the 5 interior angles are equal each interior angle is equal to $\frac{540^{\circ}}{5} = 108^{\circ}$.

Note: By drawing lines from one vertex to each of the others in a polygon, you subdivide a polygon with 4 sides (a quadrilateral) into 2 triangles, a polygon with 5 sides (a pentagon) into 3 triangles, and, in general, a polygon with n sides (an "n-gon") into $n - 2$ triangles. The sum of the interior angles of an n-sided polygon is equal to $180(n - 2)$ degrees.

Common error: Dividing $180(n-2)$ degrees by n to determine the size of an interior angle of an n-sided polygon. This is only correct when we know that all the n interior angles of the n-sided polygon are equal, that is, the polygon is "regular".

Practice: **(**i) Find the sum of the interior angles of a polygon with 9 sides.
(ii) Find the interior angle of a regular 9-gon.

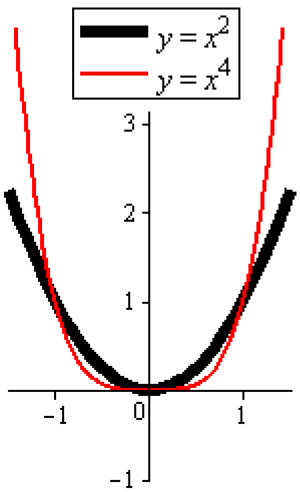
Answers: **(**i) 1260° (ii) 140°

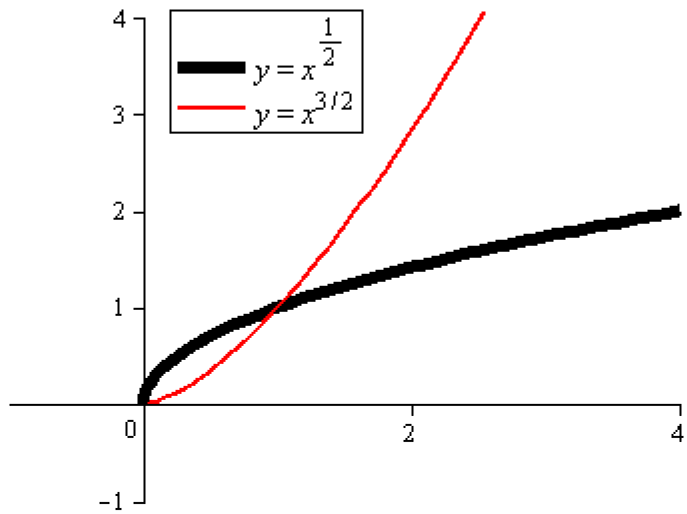
A Great Video for More Details: https://www.youtube.com/watch?v=caspFRZfWeI

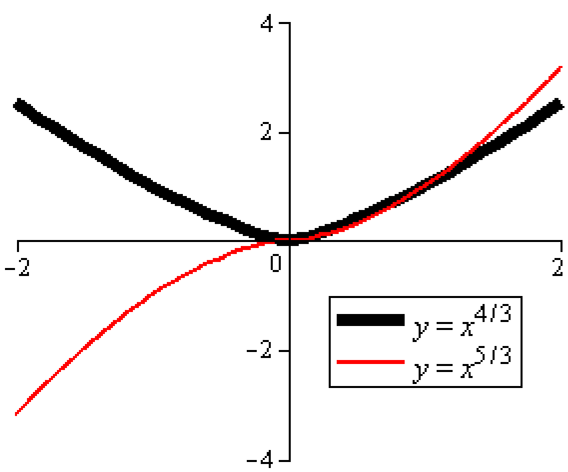
A Great Website for More Detail: https://www.mathsisfun.com/geometry/interior-angles-polygons.html

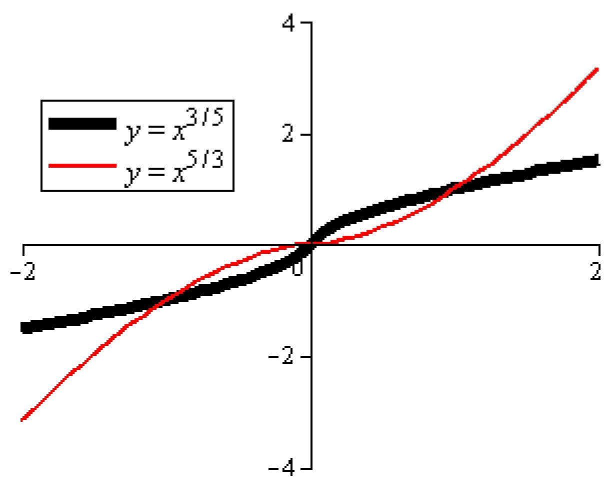
Solutions Part 5

BASIC GRAPHS





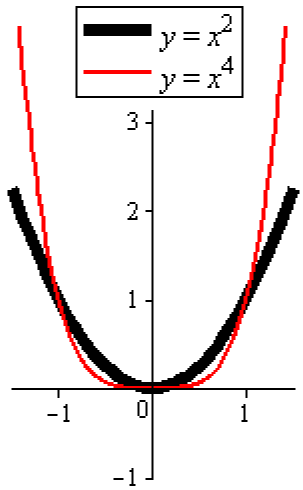




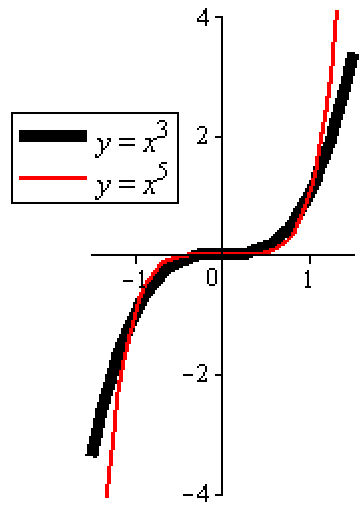
1) Graphing $y = x^n$, $n \in \mathbb{N}$

Problem: (i) Graph the curves $y = x^2$ and $y = x^4$ on the same set of axes.
(ii) Graph the curves $y = x^3$ and $y = x^5$ on the same set of axes.

Solution: (i)



(ii)

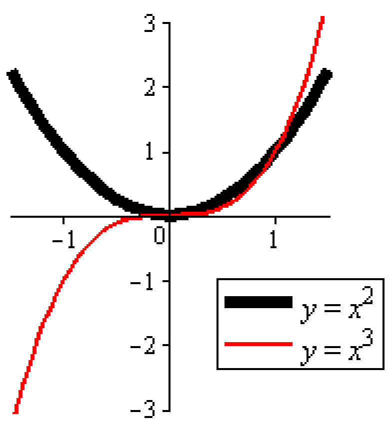


Notes: When n is even, $y = x^n$ is symmetric in the y axis. That is why graphs symmetric in the y axis are called **even functions.** When n is odd, $y = x^n$ is symmetric in the origin. That is why graphs symmetric in the origin are called **odd functions.**

Common error : Most of us realize $y = x^4$ is above $y = x^2$ when $x > 1$ but do not realize it is BELOW $y = x^2$ when $0 < x < 1$.

Practice: Graph $y = x^2$ and $y = x^3$ on the same set of axes..

Answer:

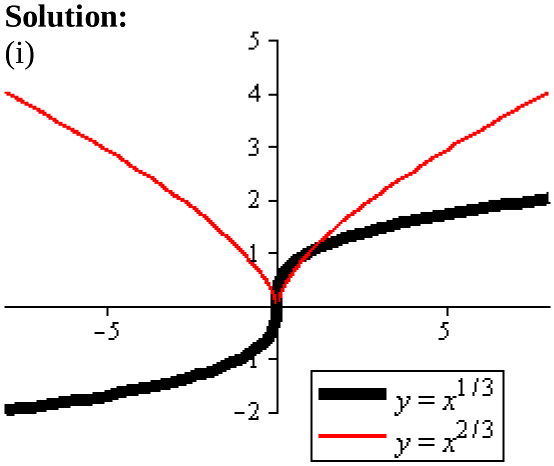


A Great Video for More Detail: https://www.youtube.com/watch?v=WKKFftAL4aI
A Great Website for More Detail: https://www.desmos.com/calculator/jfesw7c1re

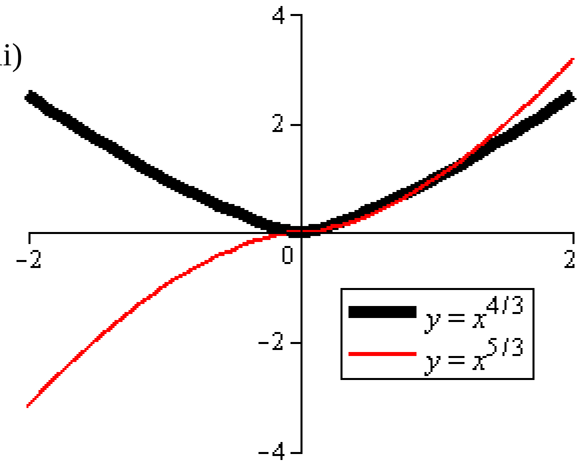
2) Graphing $y = x^{m/n}$, $m, n \in \mathbb{N}$, n Odd, and m/n is a Reduced Fraction

Problem: (i) Graph the curves $y = x^{1/3}$ and $y = x^{2/3}$ on the same set of axes.
(ii) Graph the curves $y = x^{4/3}$ and $y = x^{5/3}$ on the same set of axes.

Solution:
(i)



(ii)

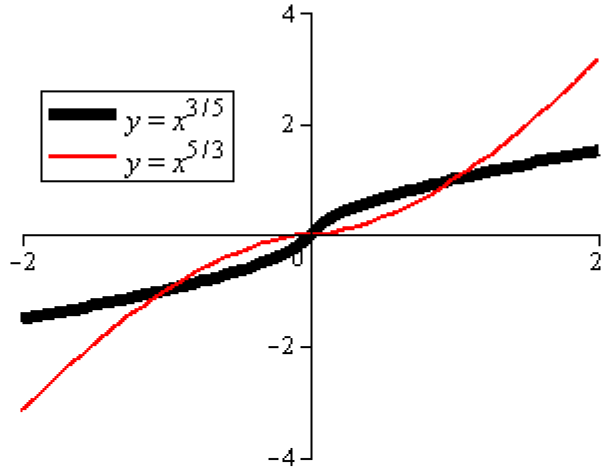


Notes: When $0 < m/n < 1$, the graph of $y = x^{m/n}$ looks like the square root or cube root function. When $m/n > 1$, the graph of $y = x^{m/n}$ looks like the parabola or the cubic.

Common error: Mixing up when $y = x^{m/n}$ is always positive and when it is sometimes + and sometimes $-$.

Practice: Graph $y = x^{3/5}$ and $y = x^{5/3}$ on the same set of axes..

Answer:

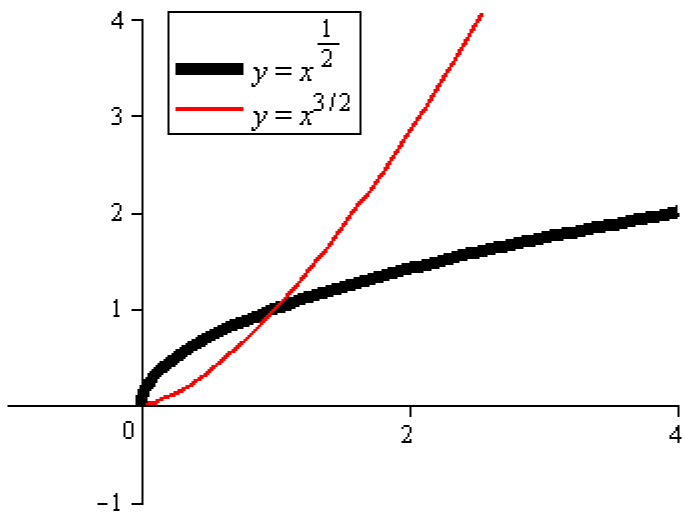


A Great Video for More Detail: **https://www.youtube.com/watch?v=9Giu9tk6H6I**
A Great Website for More Detail: **https://www.emathhelp.net/notes/algebra-2/trigonometry/power-function-with-positive-fractional-exponent/**

3) Graphing $y = x^{m/n}$, $m, n \in \mathbb{N}$, n Even, and m/n is a Reduced Fraction

Problem: Graph the curves $y = x^{1/2}$ and $y = x^{3/2}$ on the same set of axes.

Solution:

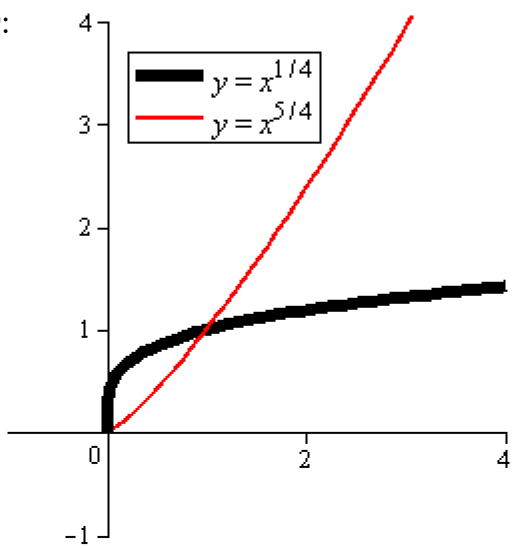


Notes: The domain of both of these functions is $\{x \in \mathbb{R} \mid x \geq 0\}$. When the exponent is between 0 and 1, the graph is concave down. When it is greater than 1, it is concave up.

Common error: Forgetting that x must be ≥ 0.

Practice: Graph $y = x^{1/4}$ and $y = x^{5/4}$ on the same set of axes..

Answer:

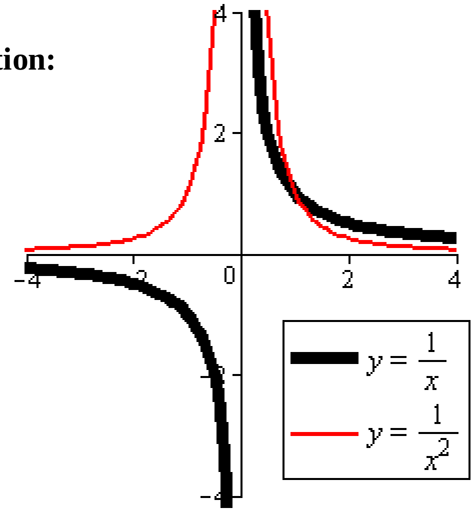


A Great Video for More Detail: **https://www.youtube.com/watch?v=9Giu9tk6H6I**
A Great Website for More Detail: **https://www.emathhelp.net/notes/algebra-2/trigonometry/power-function-with-positive-fractional-exponent/**

4) Graphing $y = x^{-n} = \dfrac{1}{x^n}, n \in \mathbb{N}$

Problem: Graph the curves $y = x^{-1} = \dfrac{1}{x}$ and $y = x^{-2} = \dfrac{1}{x^2}$ on the same set of axes.

Solution:



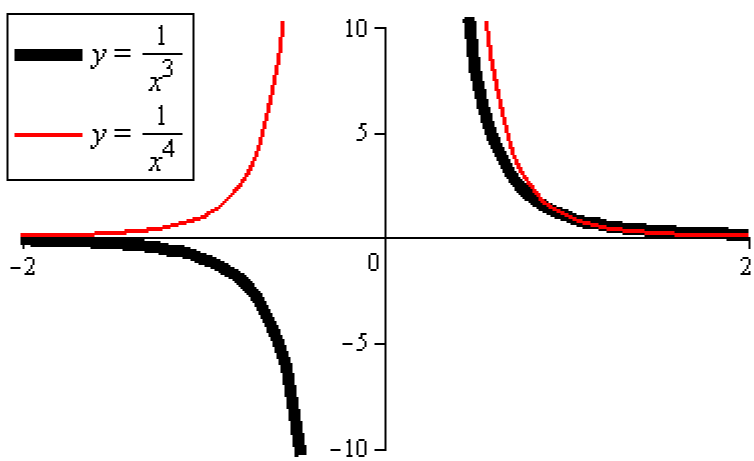
Notes: First, we know $x \neq 0$. When n is even, $y = x^{-n} = \dfrac{1}{x^n} > 0$.

When n is odd, $y = x^{-n} = \dfrac{1}{x^n} > 0$ when $x > 0$ and $y = x^{-n} = \dfrac{1}{x^n} < 0$ when $x < 0$.

Common error: Confusing $y = x^{-n} = \dfrac{1}{x^n}$ with $y = x^{1/n}$(which I — the author — did twice while composing this darn question!)

Practice: Graph $y = x^{-3} = \dfrac{1}{x^3}$ and $y = x^{-4} = \dfrac{1}{x^4}$ on the same set of axes..

Answer:

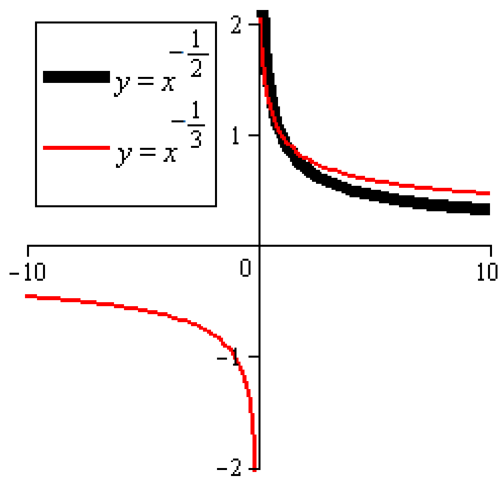


A Great Video for More Detail: https://www.youtube.com/watch?v=9Giu9tk6H6I
A Great Website for More Detail: https://www.emathhelp.net/notes/algebra-2/trigonometry/power-function-with-positive-fractional-exponent/

5) Graphing $y = x^{-\frac{1}{n}} = \dfrac{1}{x^{\frac{1}{n}}}$, $n \in \mathbb{N}$

Problem: Graph the curves $y = \dfrac{1}{x^{1/2}}$ and $y = \dfrac{1}{x^{1/3}}$ on the same set of axes.

Solution:



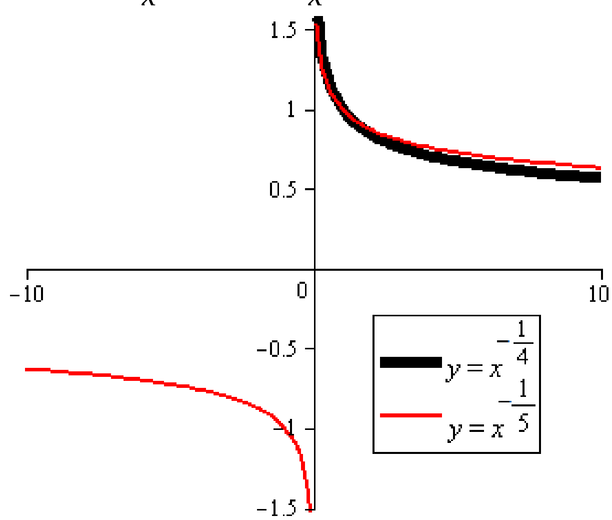
Notes: First, we know $x \neq 0$. When n is even, $y = \dfrac{1}{x^{1/n}} > 0$.

When n is odd, $y = \dfrac{1}{x^{1/n}} > 0$ when $x > 0$ and $y = \dfrac{1}{x^{1/n}} < 0$ when $x < 0$.

Common error: Confusing $y = x^{-n} = \dfrac{1}{x^n}$ with $y = \dfrac{1}{x^{1/n}}$

Practice: Graph $y = \dfrac{1}{x^{1/4}}$ and $y = \dfrac{1}{x^{1/5}}$ on the same set of axes..

Answer:

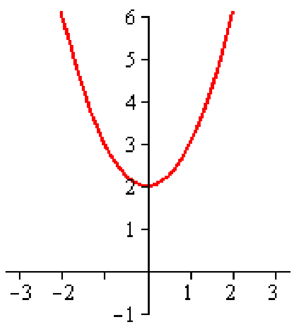


A Great Website for More Detail: Apologies. We couldn't find one we liked.
An Okay Website for More Detail:
wmueller.com/precalculus/families/pwrfrac.html

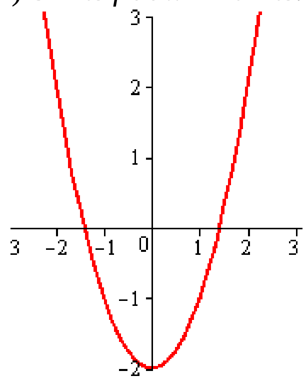
6) Transformations (New Graphs from a Given Graph)

Problem: Given $y = f(x) = x^2$, graph and describe each of the following relative to *f*.
(i) $y = f(x) + 2$ (ii) $y = f(x) - 2$ (iii) $y = f(x+2)$ (iv) $y = f(x-2)$ (v) $y = f(2x)$ (vi) $y = 2f(x)$

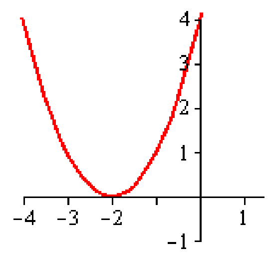
Solution:
(i) Shifts f up 2 units.



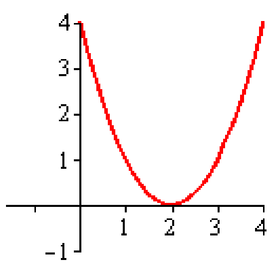
(ii) Shifts f down 2 units.



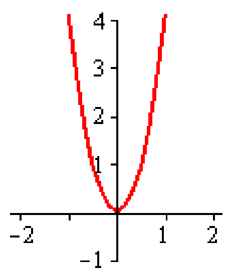
(iii) Shifts f left 2 units.



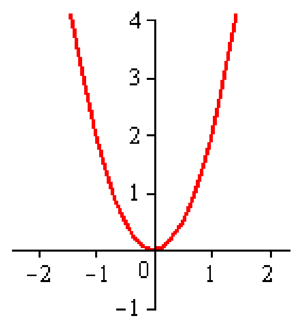
(iv) Shifts f right 2 units.



(v) Squeezes f by a factor of 2 units.



(vi) Stretches f by a factor of 2 units.



Note: Use this for graphs of every variety—trig, logs, exponents, **more!**

Common error: Thinking $f(x+2)$ shifts 2 units to the right because we added 2.

Practice: Given $y = f(x) = \sin(x)$, describe each of the following relative to *f*.
(i) $y = f(x+\pi)$ (ii) $y = f(\pi x)$ (iii) $y = f(x) + \pi$
(iv) $y = f(x-\pi)$ (v) $y = \pi f(x)$ (vi) $y = f(x) - \pi$

Answer: (i) Shifts f left π units. (ii) Squeezes f by a factor of π.
(iii) Shifts f up π units. (iv) Shifts f right π units. (v) Stretches f by a factor of π units.
(vi) Shifts f down π units.

A Great Video for More Detail:
https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:transformations/x2ec2f6f830c9fb89:trans-all-together/v/shifting-and-reflecting-functions
A Great Website for More Detail:
http://tutorial.math.lamar.edu/Classes/Alg/Transformations.aspx

Solutions Part 6

Solving

$$ax + b = 0 \Leftrightarrow x = -\frac{b}{a}$$

$$ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$ax^3 + bx^2 + cx + d = 0 \Leftrightarrow x = \text{ It gets complicated!}$$

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \Leftrightarrow$$
$$x = \text{ It gets even more complicated!}$$

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \Leftrightarrow$$
$$x = \text{ There is no formula!}$$

Equations

1) Solving Linear Equations in One Variable

Problem: Solve each of the following equations:

(i) $4x+20=2-5x$ (ii) $8(x-4)+x=6(x-5)-(1-x)$ (iii) $\dfrac{x}{3}-\dfrac{2x}{5}+\dfrac{1}{30}=\dfrac{7}{10}$

Solution: (i) $4x+20=2-5x \quad \overset{\boxed{\text{Bring } x \text{ terms to the left side and constants to the right.}}}{\Leftrightarrow} \quad 9x=-18 \quad \overset{\boxed{\text{Divide both sides by the coefficient of } x.}}{\Leftrightarrow} \quad x=-2$

(ii) $8(x-4)+x=6(x-5)-(1-x) \quad \overset{\boxed{\text{Expand and simplify each side.}}}{\Leftrightarrow} \quad 9x-32=7x-31$

$\overset{\boxed{\text{Bring } x \text{ terms to the left side and constants to the right.}}}{\Leftrightarrow} \quad 2x=1 \quad \overset{\boxed{\text{Divide both sides by the coefficient of } x.}}{\Leftrightarrow} \quad x=\dfrac{1}{2}$

(iii) $\dfrac{x}{3}-\dfrac{2x}{5}+\dfrac{1}{30}=\dfrac{7}{10} \quad \overset{\boxed{\text{Multiply the left side and the right side by the } \textbf{least common multiple} \text{ of } 3, 5, \text{ and } 10, \text{ which is } 30.}}{\Leftrightarrow} \quad 10x-12x+1=21$

$\overset{\boxed{\text{Bring } x \text{ terms to the left side and constants to the right.}}}{\Leftrightarrow} \quad -2x=20 \quad \overset{\boxed{\text{Divide both sides by the coefficient of } x.}}{\Leftrightarrow} \quad x=-10$

Note: Here is the cardinal rule of high school math:

WYDTOSYDTTO! $\equiv$ WHAT YOU DO TO ONE SIDE YOU DO TO THE OTHER!

Common error: Messing up one of the cardinal rules of math: “What you do to one side, you do to the other!”

Practice: Solve each of the following equations:

(i) $7(4x-5)=8(3x-5)+9$ (ii) $\dfrac{x-1}{2}+\dfrac{2x+1}{5}=6$

Answer: (i) $x=1$ (ii) $x=7$

A Great Video for More Details:
https://www.youtube.com/watch?v=gSWTqZrC7Ac
A Great Website for More Details:
wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut14_lineareq.htm

2) Solving Linear Inequalities

Problem: Solve the following inequalities:
(i) $4x-5\le 2x+9$ (ii) $2x+7>5x-1$ (iii) $x-4<x+6$

Solution: (i) $4x-5\le 2x+9 \overset{\boxed{\substack{\text{Bring } x \text{ terms to the left side}\\ \text{and constants to the right.}}}}{\Leftrightarrow} 2x\le 14 \overset{\boxed{\substack{\text{Divide both sides by}\\ \text{the coefficient of } x.}}}{\Leftrightarrow} x\le 7 \;\therefore\; x\in[7,\infty)$

(ii) $2x+7>5x-1 \overset{\boxed{\substack{\text{Bring } x \text{ terms to the left side}\\ \text{and constants to the right.}}}}{\Leftrightarrow} -3x>-8$

$\overset{\boxed{\substack{\text{Divide both sides by}\\ \text{the coefficient of } x.}}}{\Leftrightarrow} x \overset{\boxed{\substack{\textbf{When you } \times \textbf{ or } \div \textbf{ an inequality by a } -,\\ \textbf{the direction of the inequality REVERSES!}}}}{<} \dfrac{3}{8} \;\therefore\; x\in\left(-\infty,\dfrac{3}{8}\right)$

(iii) $x-4<x+6 \Leftrightarrow 0<10$ This is always true! $\therefore\; x\in\mathbb{R}$

Note: Solve linear inequalities just as you would a linear equation, except when you multiply or divide by a "–". Then you must change "<" to ">" and vice-versa!

Common error: Not paying attention to the note!

Practice: Solve the following inequalities: (i) $5(x-3)\le 3(x-4)$ (ii) $4x-3<-7+4x$

Answer: (i) $x\in\left(-\infty,\dfrac{3}{2}\right)$ (ii) No solution (ie., ϕ)

A Great Video for More Details:
https://www.youtube.com/watch?v=6SzLLRDnJYE
A Great Website for More Details: purplemath.com/modules/ineqlin.htm

3) Solving Two Linear Equations in Two Variables

Problem: Solve for x and y. (E1 and E2 refer to equation 1 and equation 2.)
(i) E1: $x+2y=-1$ (ii) E1: $x-2y=6$ (iii) E1: $x-2y=6$
E2: $5x-2y=7$ E2: $3x-6y=18$ E2: $3x-6y=3$

Solution: (i) Add E1 + E2: $6x=6 \Leftrightarrow x=1$ Substitute $x=1$ into E1:
$1+2y=-1 \Leftrightarrow 2y=-2 \Leftrightarrow y=-1$ $\therefore$ The solution is $(x,y)=(1,-1)$.

(ii) Multiply $3\times \text{E1}=\text{E3}$: $3x-6y=18$
Subtract E3 – E2: $0=0$ This is ALWAYS true! From E1: $x=6+2y$
$\therefore$ Solutions are $\{(6+2y, y) \mid y\in\mathbb{R}\}$

(iii) Multiply $3\times \text{E1}=\text{E3}$: $3x-6y=18$
Subtract E3 – E2 = E4: $0=15$ This is NEVER true!
$\therefore$ There are no solutions.

Notes: There are LOTS of ways to solve these problems. Each equation represents a line. In (i), the lines are non-parallel and meet in a single point. In (ii), the lines are "coincident". In (iii), they are parallel and never meet.

Common error: Making an arithmetic mistake when forming a new equation from the given equations.

Practice: Solve $3x-2y=7$ and $2x-5y=12$

Answer: $(x,y)=(1,-2)$

A Great Video for More Details:
httpshttps://www.youtube.com/watch?v=oKqtgz2eo-Y
A Great Website for More Details:
wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut49_systwo.htm

4) Solving Quadratic Equations

Problem: Solve the following quadratic equations:
(i) $x^2 - 5x + 6 = 0$ (ii) $3x^2 - 7x + 2 = 0$

Solution: (i) $x^2 - 5x + 6 = 0 \Leftrightarrow (x-2)(x-3) = 0 \Leftrightarrow x = 2$ or $x = 3$

(ii) $3x^2 - 7x + 2 = 0$

Here, it is easiest to use the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x \overset{\boxed{a=3,\ b=-7,\ c=2}}{=} \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(2)}}{2(3)} = \frac{7 \pm \sqrt{25}}{6}$$

$$\therefore\ x = \frac{12}{6} = 2 \quad \text{or} \quad x = \frac{2}{6} = \frac{1}{3}$$

Note: In (ii), if you factored, you would have found $3x^2 - 7x + 2 = (3x-1)(x-2) = 0$, which again (thank the math gods) gives $x = \dfrac{1}{3}$ or $x = 2$.

Common error: Factoring incorrectly.

Practice: Solve the quadratic equations: (i) $x^2 - 3x - 10 = 0$ (ii) $6x^2 - 7x - 3 = 0$

Answer: (i) $x = -2, 5$ (ii) $x = -\dfrac{1}{3}, \dfrac{3}{2}$

A Great Video for More Details:
https://www.youtube.com/results?search_query=solving+quadratic+equations
A Great Website for More Details: **purplemath.com/modules/solvquad.htm**

5) Solving Equations Involving Square Roots

Problem: Solve the following equations for *x*:

(i) $\sqrt{x-2}=5$ (ii) $\sqrt{4-3x}=x+12$ (iii) $\sqrt{1+2x}-\sqrt{x}=1$

Solution: (i) $\sqrt{x-2}=5 \xRightarrow{\boxed{\text{Square both sides.}}} x-2=25 \Rightarrow x=27$

Check: Substitute $x=27$ in the original equation:

Left Side $=\sqrt{27-2}=\sqrt{25}=5$ Right Side $=5$ $\therefore x=5$

(ii) $\sqrt{4-3x}=x+12 \xRightarrow{\boxed{\text{Square both sides.}}} 4-3x=x^2+24x+144$

$\xRightarrow{\boxed{\text{Move "everything" to one side and solve.}}} x^2+27x+140=0 \Rightarrow (x+20)(x+7)=0 \Rightarrow x=-20$ or $x=-7$

Check: Substitute $x=-20$ in the original equation:

Left Side $=\sqrt{4+60}=8$ Right Side $=-20+12=-8$ $\therefore x=-20$ is NOT a solution.

Check: Substitute $x=-7$ in the original equation:

Left Side $=\sqrt{4+21}=5$ Right Side $=-7+12=5$ $\therefore x=-7$ is the only solution.

(iii) $\sqrt{1+2x}-\sqrt{x}=1 \xRightarrow{\boxed{\text{Isolate the more complicated square root on one side.}}} \sqrt{1+2x}=1+\sqrt{x} \xRightarrow{\boxed{\text{Square both sides.}}} 1+2x=1+2\sqrt{x}+x$

$\xRightarrow{\boxed{\text{Isolate the remaining square root.}}} 2\sqrt{x}=x \xRightarrow{\boxed{\text{Square both sides.}}} 4x=x^2 \xRightarrow{\boxed{\text{Move "everything" to one side and solve.}}} x^2-4x=x(x-4)=0 \therefore x=0$ or $x=4$

Check: Exercise for you but both $x=0$ and $x=4$ work!

Note: When we square both sides of an equation, we may introduce solutions to the new equation that are **not** solutions to the original. For example, $x=-3 \Rightarrow x^2=9$. The new equation also has $x=3$ as a solution, but this is not a solution of the original equation, $x=-3$. By squaring, we introduced an "extraneous" solution.

Common error: Forgetting to do a left side/right check side for extraneous roots.

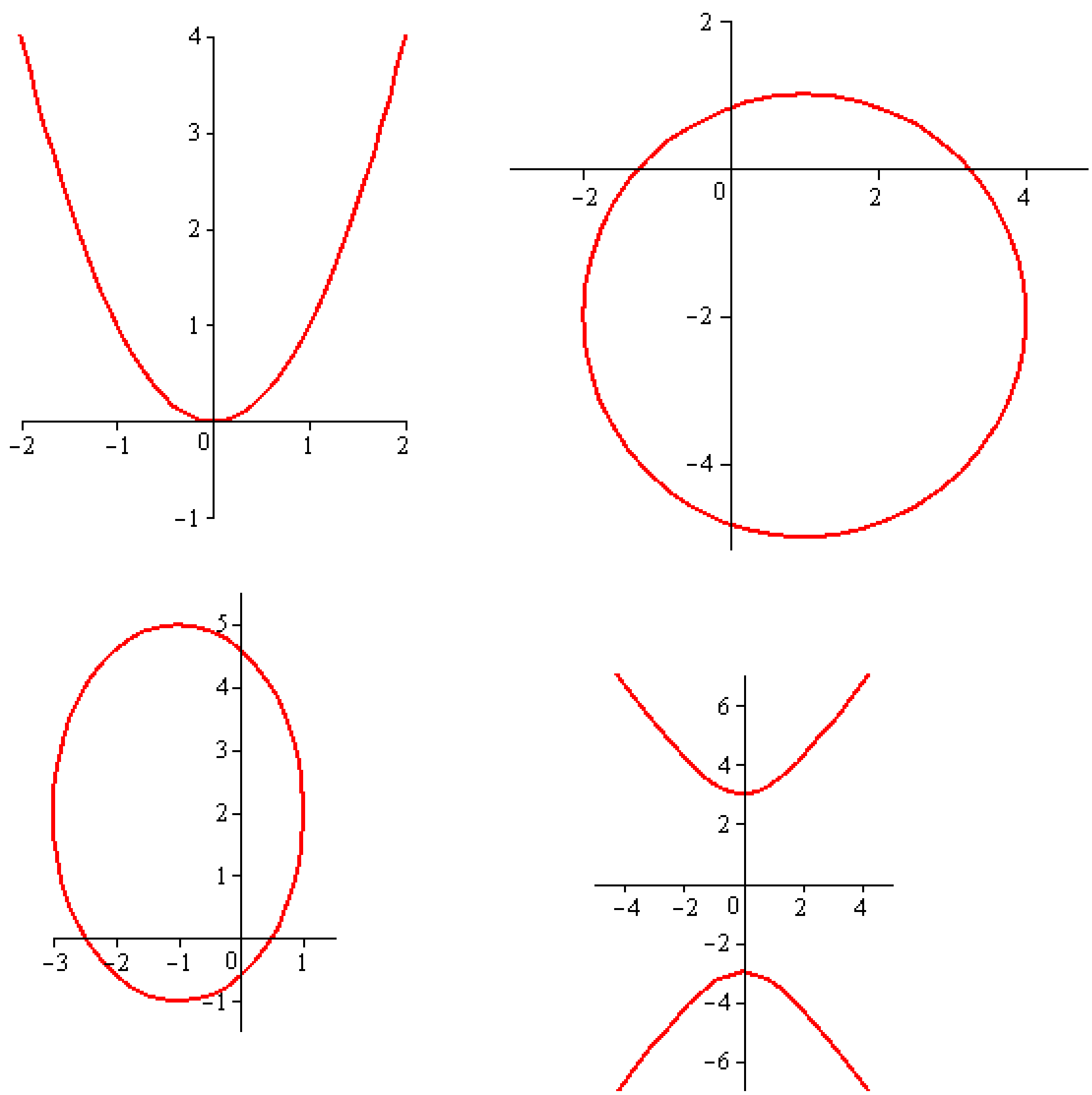
Practice: Solve the following equations: (i) $\sqrt{x+4}=6$ (ii) $\sqrt{x-3}=x-5$.

Answer: (i) $x=32$ (ii) $x=7$

A Great Video for More Details: https://www.youtube.com/watch?v=0gicD4STzpg
A Great Website for More Detail: purplemath.com/modules/solverad.htm

Solutions Part 7

GRAPHING SECOND

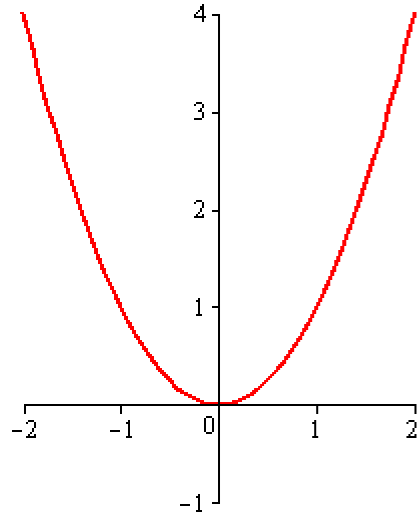


ORDER RELATIONS

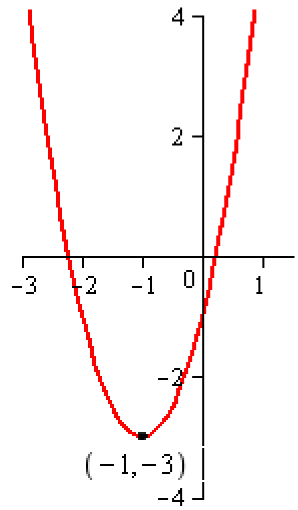
1) The Parabola

Problem: Graph the following parabolas and identify the vertex and the axis of symmetry: (i) $y = x^2$ (ii) $y = 2(x+1)^2 - 3$

Solution: (i)



(ii)



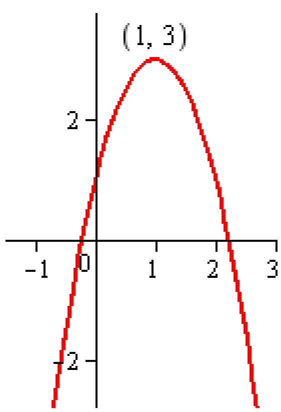
Vertex: $(0,0)$; Axis of Symmetry: $x = 0$ Vertex: $(-1,-3)$; Axis of Symmetry: $x = -1$

Note: In $y = a(x-b)^2 + c,$ the vertex is (b,c) and the axis of symmetry is $x = b.$
When $a > 0,$ the parabola opens up; when $a < 0,$ the parabola opens down.

Common error: In (ii), identifying the vertex as $(1,-3)$.

Practice: Graph $y = -2(x-1)^2 + 3$ and identify the vertex and the axis of symmetry.

Answer:



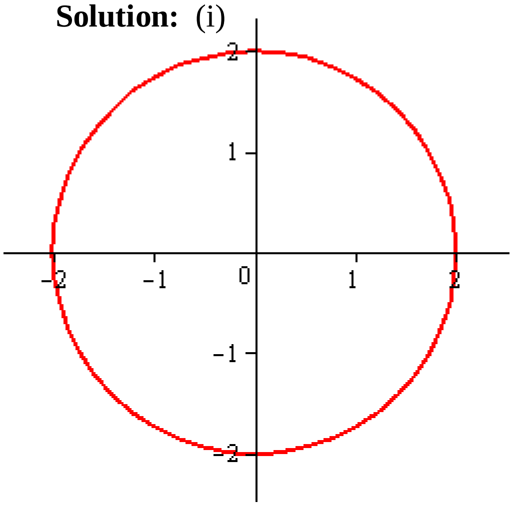
Vertex: $(1,3)$
Axis of Symmetry: $x = 1$

A Great Video for More Detail: https://www.youtube.com/watch?v=Hq2Up_1Ih5E
A Great Website for More Detail:
tutorial.math.lamar.edu/Classes/Alg/Parabolas.aspx

2) The Circle

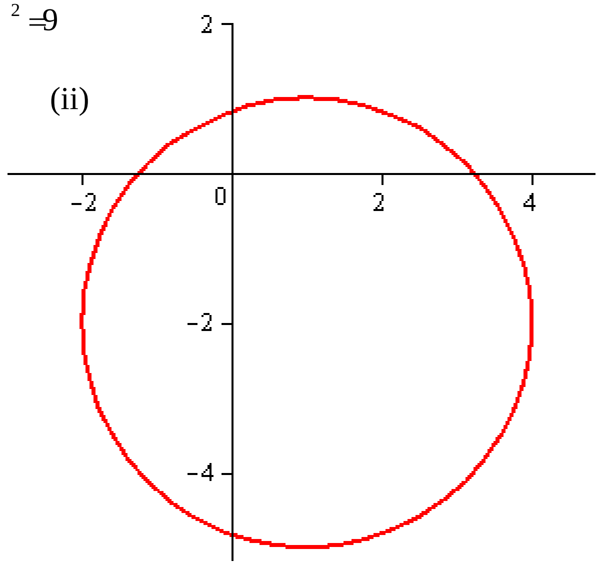
Problem: Graph the following circles and identify the radius and the centre:
(i) $x^2 + y^2 = 4$ (ii) $(x-1)^2 + (y+2)^2 = 9$

Solution: (i)



$r = 2$; Centre$(0,0)$

(ii)



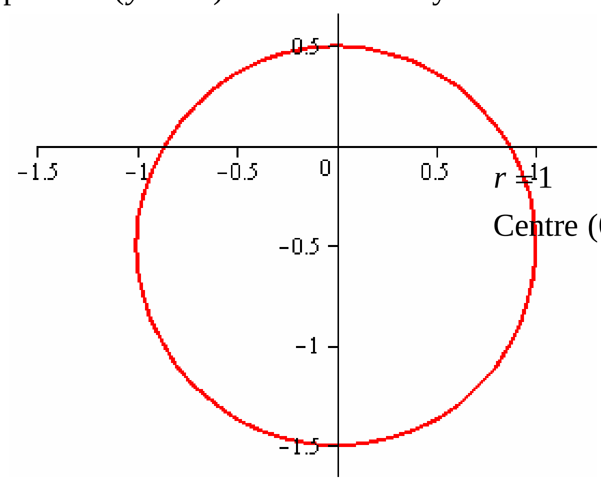
$r = 3$; Centre$(1,-2)$

Note: In $(x-a)^2 + (y-b)^2 = r^2$, the centre is (a,b) and the radius, of course, is r.

Common error: In (ii), identifying the centre as $(-1,2)$.

Practice: Graph $x^2 + (y+0.5)^2 = 1$ and identify the radius and the centre.

Answer:



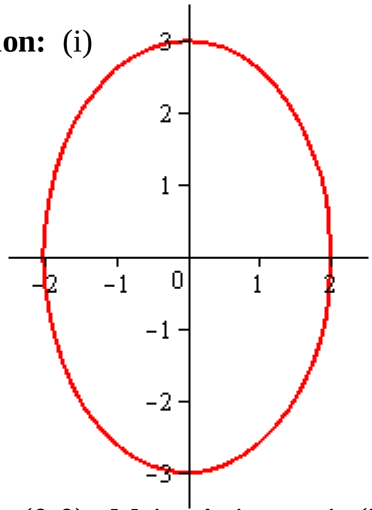
$r = 1$
Centre $(0,-0.5)$

A Great Video for More Detail: **https://www.youtube.com/watch?v=u_39J-syjB0**
A Great Website for More Detail:
wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut29_circles.htm

3) The Ellipse

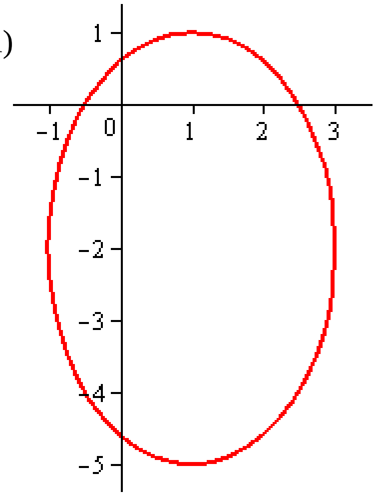
Problem: Graph the following ellipses and identify the centre and the major and minor axes: (i) $\frac{x^2}{4}+\frac{y^2}{9}=1$ (ii) $\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$

Solution: (i)



Centre $(0,0)$; Major Axis: y axis (ie., $x=0$)
Minor Axis: x axis (ie., $y=0$)

(ii)



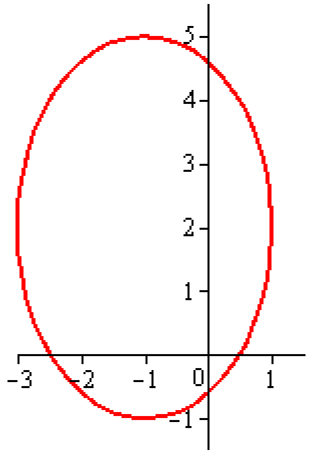
Centre $(1,-2)$; Major Axis: $x=1$
Minor Axis: $y=-2$

Note: In $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the x intercepts are $\pm a$ and the y intercepts are $\pm b$.

Common error: Confusing the major and minor axes.

Practice: Graph $\frac{(x+1)^2}{4}+\frac{(y-2)^2}{9}=1$ and identify the centre and the major and minor axes:

Answer:



Centre $(-1,2)$
Major Axis: $x=-1$; Minor Axis: $y=2$

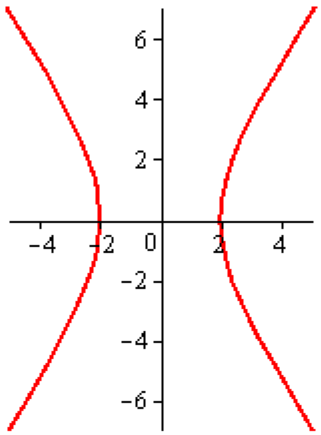
A Great Video for More Detail: https://www.youtube.com/watch?v=HvV46ka3vZ0
A Great Website for More Detail:
algebralab.org/lessons/lesson.aspx?file=Algebra_conics_ellipse.xml

4) The Hyperbola

Problem: Graph the following hyperbolas and identify the centre and intercepts:

(i) $\frac{x^2}{4}-\frac{y^2}{9}=1$ (ii) $\frac{y^2}{9}-\frac{x^2}{4}=1$

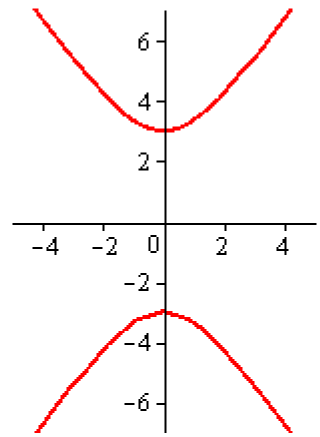
Solution: (i)



Centre $(0,0)$

x intercepts $=\pm 2$; no y intercepts

(ii)



Centre $(0,0)$

no x intercepts; y intercepts $=\pm 3$

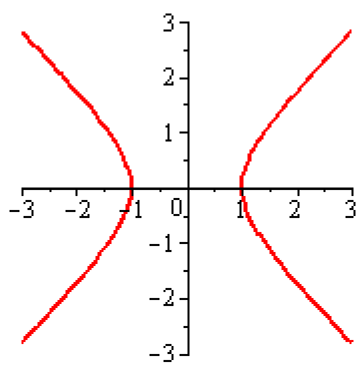
Note: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a hyperbola opening on the x axis.

$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ is a hyperbola opening on the y axis.

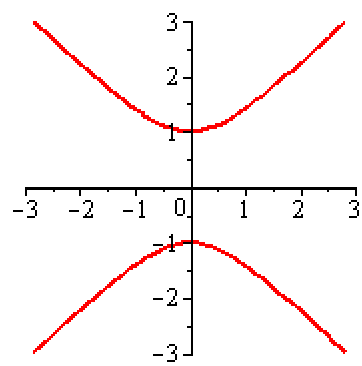
Common Error: Putting the intercepts on the wrong axis.

Practice: Graph (i) $x^2-y^2=1$ (ii) $y^2-x^2=1$.

Answer: (i)



(ii)



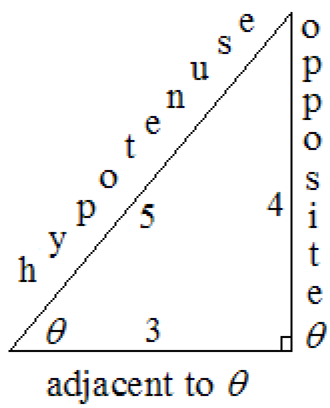
A Great Video for More Detail: **https://www.youtube.com/watch?v=I9FU2IrZoiA**

A Great Website for More Detail:

tutorial.math.lamar.edu/Classes/Alg/Hyperbolas.aspx

Solutions Part 8

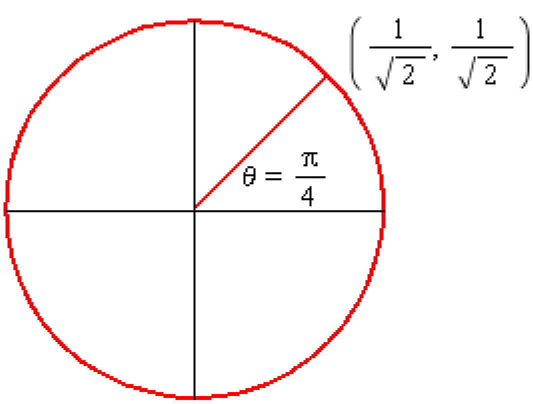
TRIGONOMETRY

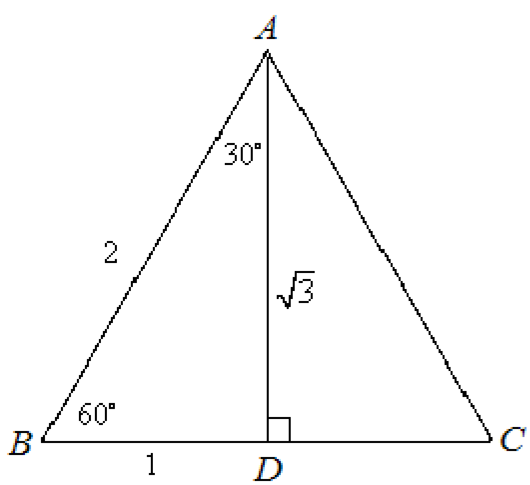


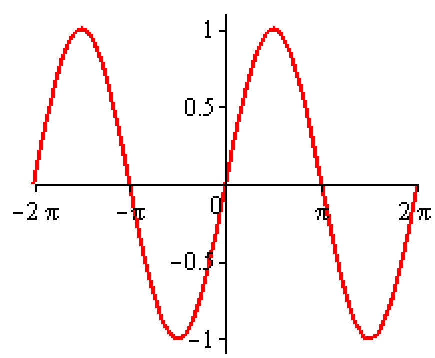
S $\boxed{\text{SIN}}$ O $\boxed{\text{OPPOSITE}}$ H $\boxed{\text{HYP0TENUSE}}$ **C** $\boxed{\text{COS}}$ A $\boxed{\text{ADJACENT}}$ H $\boxed{\text{HYP0TENUSE}}$ **T** $\boxed{\text{TAN}}$ O $\boxed{\text{OPPOSITE}}$ A $\boxed{\text{ADJACENT}}$

$$\sin(\theta) = \frac{O}{H} = \frac{4}{5} \qquad \cos(\theta) = \frac{A}{H} = \frac{3}{5} \qquad \tan(\theta) = \frac{O}{A} = \frac{4}{3} = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{5}{4} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{5}{3} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{3}{4} = \frac{\cos(\theta)}{\sin(\theta)}$$





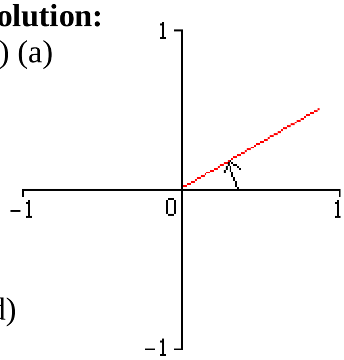


1) Angles in Standard Position: Degree Measure

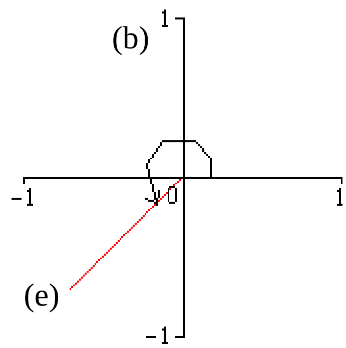
Problem: (i) Draw in standard position the following angles:
(a) 30° (b) 225° (c) -80° (d) -190° (e) 390°

(ii) Give all angles, in degrees, which are "co-terminal" with (a) 30° (b) -190°.

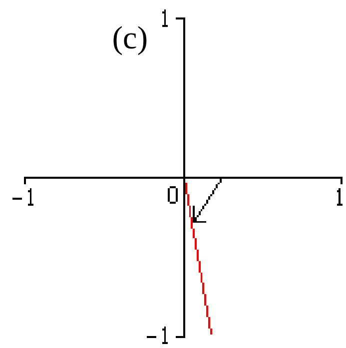
Solution:
(i) (a)



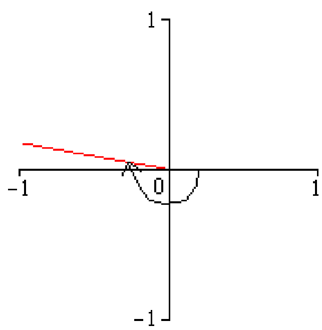
(b)



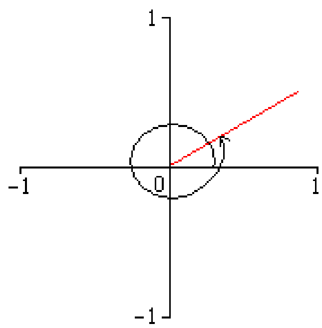
(c)



(d)



(e)



(ii) (a) All angles co-terminal with 30° are $30^\circ + 360^\circ k$, where where $k \in \mathbb{Z}$..
(b) All angles co-terminal with -190° are $-190^\circ + 360^\circ k$, where $k \in \mathbb{Z}$.

Note: Co-terminal angles differ by multiples of 360°.

Common error: Regarding co-terminal angles as equal. 30° and 390° are different angles whose terminal arms, in standard position, are the same. Here is an analogy: Consider the function $f(x) = x^2$. $f(-3) = f(3) = 9$ but that doesn't make $-3 = 3$!

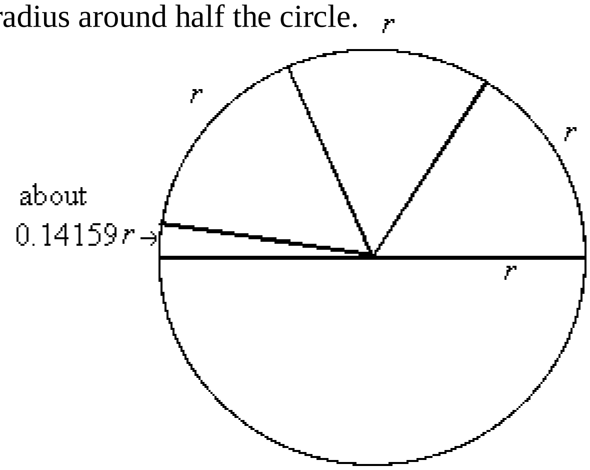
Practice: (i) In what quadrant is the terminal arm of -660°?
(ii) Give all the angles in degree measure co-terminal with -660°.

Answer: (i) First quadrant, since -660° is co-terminal with 60°
(ii) $-660^\circ + 360^\circ k$, where $k \in \mathbb{Z}$.

A Great Video for More Details:
https://www.youtube.com/watch?v=Xtk1PVRoDLQ
A Great Website for More Details: themathpage.com/aTrig/measure-angles.htm#stand

2) The Meaning of π

Problem: In the circle below with radius *r*, you can "fit" three and a "little portion more" of a radius around half the circle.



We give a name to this number of radii. (i) The name is _____.

(ii) So the length of a half circle is given by ______.

(iii) This is why the circumference is given by _____.

Solution: (i) π (ii) πr (iii) $2\pi r$

Note: The most commonly used approximations for π are $\frac{22}{7}$ or 3.14.

An approximation to π, accurate to 7 decimals, is 3.1415926.

Common problem (as opposed to error): Not having a clue that π is the number of times the radius of a circle wraps around a semi-circle!

Practice: The number π plays a role in the calculation of certain areas. For example, the surface area of a sphere of radius r is $4\pi r^2$ (4 times the area of a circle of radius r). Find the surface area of a sphere of radius 3.

Answer: 36π units^2

A Great Video for More Details: https://www.youtube.com/watch?v=4RldHTtd3O8
A Great Website for More Detail: en.wikipedia.org/wiki/Pi

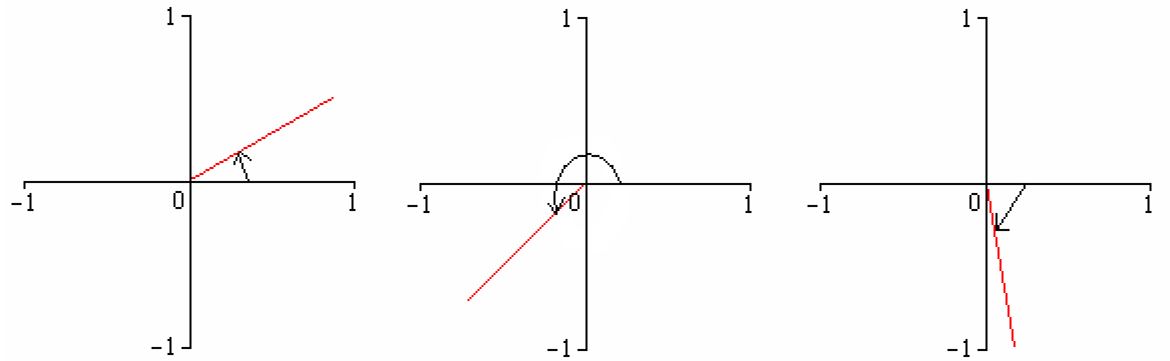
3) Angles in Standard Position: Radian Measure

Problems: 1) Draw in standard position the following angles:

(i) $\pi/6$ (ii) $\frac{5\pi}{4}$ (iii) $-\frac{4\pi}{9}$.

2) Give all angles, in radians, which are "co-terminal" with $\pi/6$ radians.

Solutions: 1)(i) (ii) (iii)



2) All angles co-terminal with $\frac{\pi}{6}$ are $\frac{\pi}{6} + 2\pi k$, where $k \in \mathbb{Z}$.

Note: 1 radian is about 57°. This is just what you should expect since a little more than 3.14 radians, that is, π radians, equals about $3.14 \times 57^\circ \doteq 180^\circ$.

Common error: The formulas for arc length and area of a sector of a circle are $r\theta$ and $\frac{1}{2}r^2\theta$, respectively. Students sometimes substitute θ in degree measure. It **must be in radians!**

Practice: (i) In what quadrant is the terminal arm of $-\frac{7\pi}{4}$?

(ii) Give all the angles in degree measure co-terminal with $-\frac{7\pi}{4}$.

Answer: (i) First quadrant, since $-\frac{7\pi}{4}$ is co-terminal with $\frac{\pi}{4}$.

(ii) $-\frac{7\pi}{4} + 2\pi k$, where $k \in \mathbb{Z}$.

A Great Video for More Details: **https://www.youtube.com/watch?v=Ndnsds-E_Lc**
A Great Website for More Details: thémathpage.com/aTrig/radian-measure.htm

4) Degrees to Radians

Problem: Express each of the following in radian measure:

(i) 25° (ii) -150° (iii) 1060° (Remember 180 degrees = π radians.)

Solution: (i) $25^\circ = \frac{\pi}{180^\circ}\times 25^\circ = \frac{5\pi}{36}$ radians

(ii) $-150^\circ = \frac{\pi}{180^\circ}\times(-150^\circ) = -\frac{5\pi}{6}$ radians

(iii) $1060^\circ = \frac{\pi}{180^\circ}\times 1060^\circ = \frac{53\pi}{9}$ radians

Note: $1^\circ = \frac{\pi}{180}$ radians $\therefore x^\circ = x\left(\frac{\pi}{180}\right)$ radians

Common misinterpretation: We always use the degree symbol when measuring with degrees but we don't always say radians when using radian measure. So, when measuring angles, "2°", "2 radians", and "2" mean, respectively, "2 degrees", "2 radians", and "2 **radians"** (which is about 115 degrees!)

Practice: Express each of the following in radian measure:
(i) -210° (ii) 300° (iii) 240°

Answer: (i) $\frac{-7\pi}{6}$ (ii) $\frac{5\pi}{3}$ (iii) $\frac{4\pi}{3}$

A Great Video for More Details: **https://www.youtube.com/watch?v=ieg_YfnMxec**
A Great Website for More Detail: **https://www.wikihow.com/Convert-Degrees-to-Radians**

5) Radians to Degrees

Problem: Express each of the following radian measures in degrees:

(i) $\frac{4\pi}{9}$ (ii) $\frac{2}{5}$ (iii) $-\frac{7\pi}{6}$ (iv) $-\frac{4\pi}{3}$ (Remember π radians = 180 degrees)

Solution: (i) $\frac{4\pi}{9}$ radians = $\frac{4\pi}{9} \times \frac{180^\circ}{\pi} = 80^\circ$.

(ii) $\frac{2}{5}$ radians = $\frac{2}{5} \times \frac{180^\circ}{\pi} = \frac{72}{\pi} = 22.92^\circ$.

(iii) $-\frac{7\pi}{6}$ radians = $-\frac{7\pi}{6} \times \frac{180^\circ}{\pi} = -210^\circ$.

(iv) $-\frac{4\pi}{3}$ radians = $-\frac{4\pi}{3} \times \frac{180^\circ}{\pi} = -240^\circ$.

Note: 1 radian = $\frac{180^\circ}{\pi} \doteq 57.3^\circ$ $\therefore$ x radians = $x\left(\frac{180^\circ}{\pi}\right)$

Common misinterpretation: We always use the degree symbol when measuring with degrees but we don't always say radians when using radian measure. So, when measuring angles, "2°", "2 radians", and "2" mean, respectively, "2 degrees", "2 radians", and "2 **radians"** (which is about 115 degrees!)

Practice: Express each of the following radian measures in degrees to the nearest degree:

(i) $\frac{4}{3}$ (ii) $\frac{3\pi}{4}$ (iii) $\frac{7\pi}{10}$ (iv) $\frac{1}{4}$

Answer: **(**i) 76° (ii) 135° (iii) 126° (iv) 14°

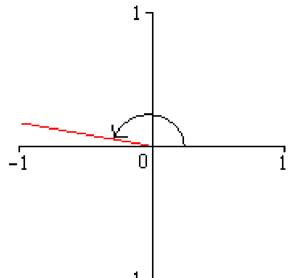
A Great Video for More Details: **https://www.youtube.com/watch?v=z0-1gBy1ykE**
A Great Website for More Details:
https://www.wikihow.com/Convert-Radians-to-Degrees

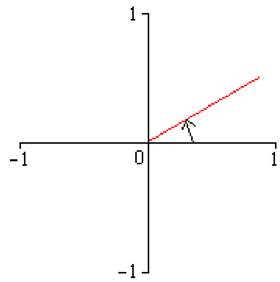
6) Relating Angles in Standard Position in Quadrants One and Two

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

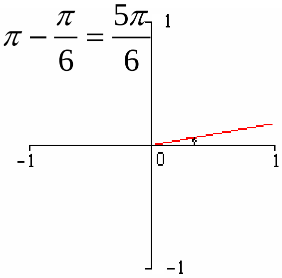
Problem: (i) Below, the second quadrant angle 170° is drawn in standard position. Find and illustrate the related first quadrant angle, using the interval $\left(0, 90^\circ\right)$.

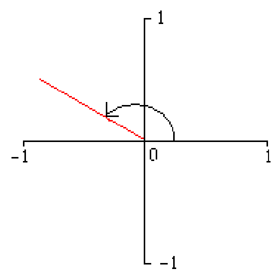
(ii) Below, the first quadrant angle $\frac{\pi}{6}$ is drawn in standard position. Find and illustrate the related second quadrant angle, using the interval $\left(\frac{\pi}{2}, \pi\right)$.





Solution: (i) First quad. angle = $180^\circ - 170^\circ = 10^\circ$ (ii) Second quad. angle = $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$





Notes: If θ satisifies $\frac{\pi}{2} < \theta < \pi$, the corresponding first quadrant angle using $\left(0, \frac{\pi}{2}\right)$ is $\pi - \theta$.

If θ satisifies $0 < \theta < \frac{\pi}{2}$, the corresponding second quadrant angle using $\left(\frac{\pi}{2}, \pi\right)$ is still $\pi - \theta$!

Common error: Confusing the second quadrant angle in standard position 170° with the 10° angle that 170° makes with the negative *x* axis.

Practice: (i) Find the first quadrant angle relatives of (a) 150° (b) $2\pi/3$.
(ii) Find the second quadrant relatives of (a) 75° (b) $\pi/9$.

Answer: (i)(a) 30° (b) $\pi/3$ (ii)(a) 105° (b) $8\pi/9$

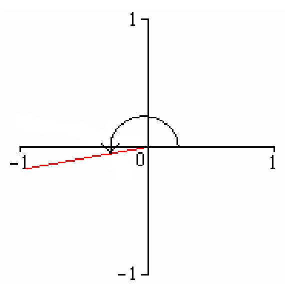
A Great Video for More Details: https://www.youtube.com/watch?v=E-xFXpVo14o
A Great Website for More Detail:
https://mathbitsnotebook.com/Algebra2/TrigConcepts/TCStandardPosition.html

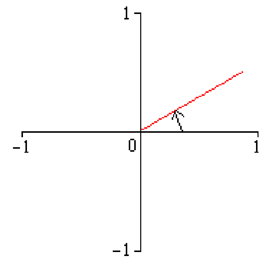
7) Relating Angles in Standard Position in Quadrants One and Three

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

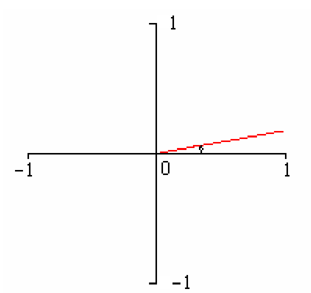
Problem: (i) Below, the third quadrant angle 190° is drawn in standard position. Find and illustrate its related first quadrant angle, using the interval $(0,90^\circ)$.

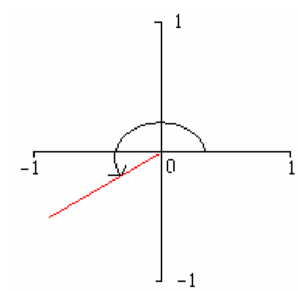
(ii) Below, the first quadrant angle $\dfrac{\pi}{6}$ is drawn in standard position. Find and illustrate its related third quadrant angle, using the interval $\left(\pi,\dfrac{3\pi}{2}\right)$.





Solution: (i) First quad. angle = $190^\circ - 180^\circ = 10^\circ$ (ii) Third quad. angle = $\pi + \dfrac{\pi}{6} = \dfrac{7\pi}{6}$





Notes: If θ satisifies $\pi < \theta < \dfrac{3\pi}{2}$, the corresponding first quadrant angle using $\left(0,\dfrac{\pi}{2}\right)$ is $\theta - \pi$.

If θ satisifies $0 < \theta < \dfrac{\pi}{2}$, the corresponding third quadrant angle using $\left(\pi,\dfrac{3\pi}{2}\right)$ is $\pi + \theta$!

Common error: Confusing the third quadrant angle in standard position 190° with the 10° angle that 190° makes with the negative *x* axis.

Practice: (i) Find the first quadrant angle relatives of (a) 250° (b) $7\pi/6$.
(ii) Find the third quadrant relatives of (a) 75° (b) $\pi/9$.

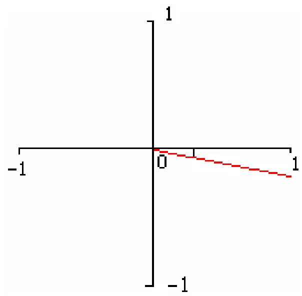
Answer: (i)(a) 70° (b) $\pi/6$ (ii)(a) 255° (b) $10\pi/9$

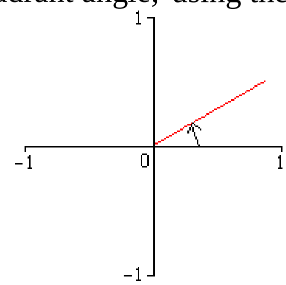
A Great Video for More Details: **https://www.youtube.com/watch?v=E-xFXpVo14o**
A Great Website for More Detail:
https://mathbitsnotebook.com/Algebra2/TrigConcepts/TCStandardPosition.html

8) Relating Angles in Standard Position in Quadrants One and Four
(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

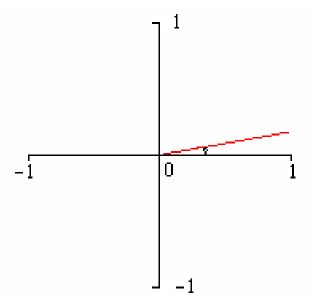
Problem: (i) Below, the fourth quadrant angle -10° is drawn in standard position. Find and illustrate its related first quadrant angle, using the interval $\left(0, 90^{\circ}\right)$.

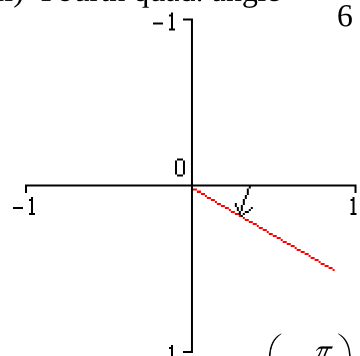
(ii) Below, the first quadrant angle $\frac{\pi}{6}$ is drawn in standard position. Find and illustrate its related fourth quadrant angle, using the interval $\left(-\frac{\pi}{2}, 0\right)$.





Solution: (i) First quad. angle = $-(-10) = 10^{\circ}$ (ii) Fourth quad. angle $= -\frac{\pi}{6}$





Notes: If $\theta \in \left(-\frac{\pi}{2}, 0\right)$, the corresponding first quadrant angle using $\left(0, \frac{\pi}{2}\right)$ is $-\theta$.

If $\theta \in \left(0, \frac{\pi}{2}\right)$, the corresponding fourth quadrant angle using $\left(-\frac{\pi}{2}, 0\right)$ is $-\theta$!

Common error: Confusing the fourth quadrant angle in standard position -10° with the 10° angle that -10° makes with the positive *x* axis.

Practice: (i) Find the **first** quadrant angle relatives of (a) -80° (b) $-5\pi/12$.
(ii) Find the **fourth** quadrant relatives of (a) 65° (b) $\pi/3$.

Answers: **(**i)(a) 80° (b) $5\pi/12$ (ii)(a) -65° (b) $-\pi/3$

A Great Video for More Details: https://www.youtube.com/watch?v=E-xFXpVo14o
A Great Website for More Detail:
https://mathbitsnotebook.com/Algebra2/TrigConcepts/TCStandardPosition.html

9) Relating an Angle in Standard Position to its "Relatives" in the other Quadrants

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

For this question, use the following restrictions:

Quadrant	Angle
1	Degrees: $0^\circ < \theta < 90^\circ$ or Radians: $0 < \theta < \frac{\pi}{2}$
2	Degrees: $90^\circ < \theta < 180^\circ$ or Radians: $\frac{\pi}{2} < \theta < \pi$: $\theta_2 \overset{\boxed{\text{degrees}}}{=} 180^\circ - \theta \overset{\boxed{\text{radians}}}{=} \pi - \theta$
3	Degrees: $180^\circ < \theta < 270^\circ$ or Radians: $\pi < \theta < \frac{3\pi}{2}$": $\theta_3 \overset{\boxed{\text{degrees}}}{=} 270^\circ - \theta \overset{\boxed{\text{radians}}}{=} \frac{3\pi}{2} - \theta$
4	Degrees: $-90^\circ < \theta < 0^\circ$ or Radians: $-\frac{\pi}{2} < \theta < 0$: $\theta_4 \overset{\boxed{\text{degrees}}}{=} -\theta \overset{\boxed{\text{radians}}}{=} -\theta$

Problem: State the standard position "relatives" of
(i) 50° (ii) 170° (iii) 250° (iv) -60° in each of the other quadrants.

Solution: Q $\equiv$ Quadrant
(i) Q2: 130°; Q3: 230°; Q4: -50° (ii) Q1: 10°; Q3: 190°; Q4: -10°
(iii) Q1: 70°; Q2: 110°; Q4: -70° (iv) Q1: 60°; Q2: 120°; Q3: 240°

Note: Remember that every angle has **LOTS** of names. Here, we were careful to specify which name we wanted to find.

Common error: You name the angle whose relatives you are trying to find using a range other than specified. For example, you might give 350° when, for the questions on this page, the answer would be -10°.

Practice: State the relatives of (i) $\frac{\pi}{4}$ (ii) $\frac{9\pi}{7}$ (iii) $\frac{13\pi}{10}$ (iv) $-\frac{\pi}{5}$.

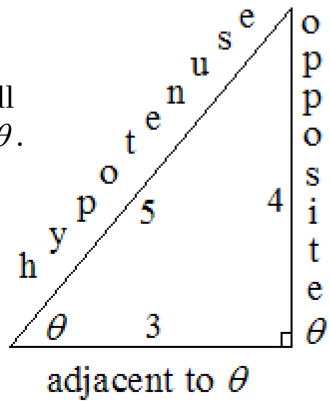
Answer: Q $\equiv$ Quadrant
(i) Q2: $\frac{3\pi}{4}$; Q3: $\frac{5\pi}{4}$; Q4: $-\frac{\pi}{4}$ (ii) Q1: $\frac{2\pi}{7}$; Q2: $\frac{5\pi}{7}$; Q4: $-\frac{2\pi}{7}$
(iii) Q1: $\frac{3\pi}{10}$; Q2: $\frac{7\pi}{10}$; Q4: $-\frac{3\pi}{10}$ (iv) Q1: $\frac{\pi}{5}$; Q2: $\frac{4\pi}{5}$; Q3: $\frac{6\pi}{5}$

A Great Video for More Details: We couldn't find one--sorry!
A Great Website for More Detail: We couldn't find one--sorry!

10) Trigonometric Ratios in Right Triangles: SOHCAHTOA

Problem:

From the triangle, identify all six trigonometric ratios for θ.



Solution:

S SIN | O OPPOSITE | H HYP0TENUSE | C COS | A ADJACENT | H HYP0TENUSE | T TAN | O OPPOSITE | A ADJACENT

$$\sin(\theta)=\frac{\text{O}}{\text{H}}=\frac{4}{5}\qquad \cos(\theta)=\frac{\text{A}}{\text{H}}=\frac{3}{5}\qquad \tan(\theta)=\frac{\text{O}}{\text{A}}=\frac{4}{3}=\frac{\sin(\theta)}{\cos(\theta)}$$

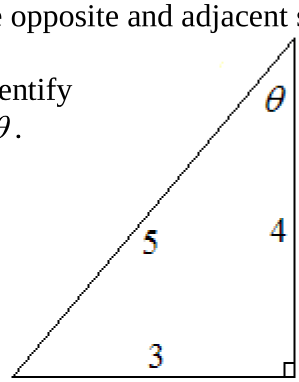
$$\csc(\theta)=\frac{1}{\sin(\theta)}=\frac{5}{4}\quad \sec(\theta)=\frac{1}{\cos(\theta)}=\frac{5}{3}\quad \cot(\theta)=\frac{1}{\tan(\theta)}=\frac{3}{4}=\frac{\cos(\theta)}{\sin(\theta)}$$

Note: These definitions only work if $0<\theta<\frac{\pi}{2}=90^{\circ}$. For other angles, we get the trig values from its "related" first quadrant angle and the CAST RULE.

(Related angles: angles whose trig ratios have the same magnitude but differ in sign according to the CAST RULE.)

Common error: Mixing up the opposite and adjacent sides.

Practice: From the triangle, identify all six trigonometric ratios for θ.



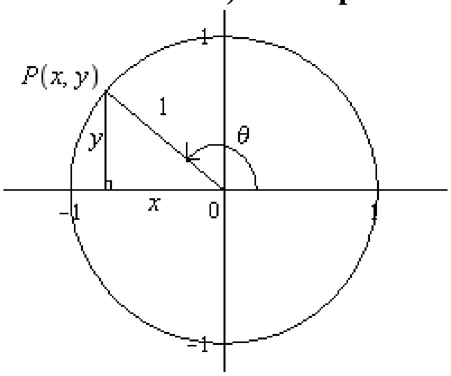
Answer: $\sin(\theta)=\frac{3}{5}\quad \cos(\theta)=\frac{4}{5}\quad \tan(\theta)=\frac{3}{4}\quad \csc(\theta)=\frac{5}{3}\quad \sec(\theta)=\frac{5}{4}\quad \cot(\theta)=\frac{4}{3}$

A Great Video for More Details: https://www.youtube.com/watch?v=5tp74g4N8EY
A Great Website for More Detail:
https://www.mathsisfun.com/algebra/sohcahtoa.html

11) Trigonometric Ratios Using the Circle: Part I

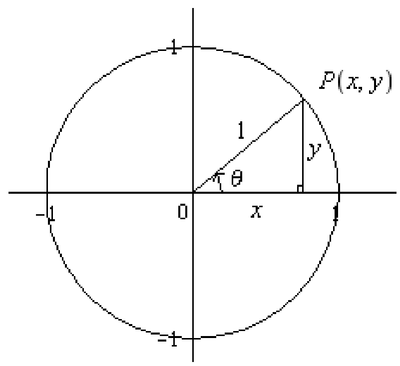
Problem: Let θ be an angle which is **not** between 0° and 90°.
By drawing the angle in standard position and letting it puncture the unit circle $x^2 + y^2 = 1$ at a point (x, y), find the sin, cos, and tan of θ.

Solution: Draw the angle in standard position. (The illustrated θ satisfies $90^\circ < \theta < 180^\circ$.) It will **puncture** the unit circle, centre the origin, at a point (x, y).



Then $\sin(\theta) = y,\ \cos(\theta) = x,$ and $\tan(\theta) = \dfrac{y}{x}.$

Note: If $0^\circ < \theta < 90^\circ$, our picture would look like this:



Using the method of the solution above, $\sin(\theta) = y, \cos(\theta) = x,$ and $\tan(\theta) = \dfrac{y}{x}.$ Using "triangle trig", $\sin(\theta) = \dfrac{O}{H} = \dfrac{y}{1} = y, \cos(\theta) = \dfrac{A}{H} = \dfrac{x}{1} = x,$ and $\tan(\theta) = \dfrac{O}{A} = \dfrac{y}{x}.$ We get the same answers. Surprised? You shouldn't be. The method of the solution is just the LOGICAL extension of trig to angles not between 0° and 90°.

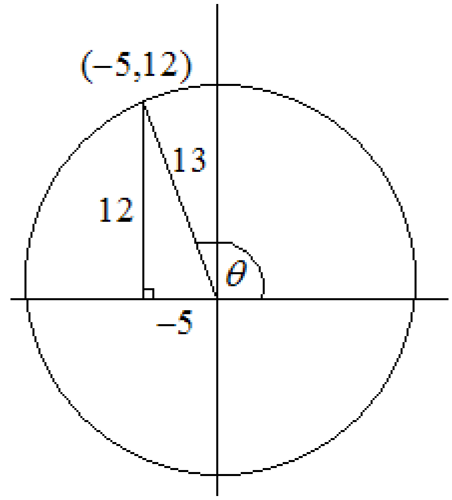
Common error: x and y will be either + or − depending on the quadrant where θ punctures the circle. Getting the "signs" wrong (and therefore often the "sin's" wrong!) is common!

Practice: If θ punctures the circle in the fourth quadrant, what are the signs of x and y and how does this relate to the CAST RULE?

Answer: x is + and y is $-$. The sin and tan will be $-$ while cos will be +.
This accounts for the **C** in the **C**AST RULE.

A Great Video for More Details:
https://www.youtube.com/watch?v=V5ArB_GFGYQ
A Great Website for more Detail: https://www.mathsisfun.com/geometry/unit-circle.html

12) Trigonometric Ratios Using the Circle: Part II



Problem: In the diagram, θ, where $90^\circ < \theta < 180^\circ$, is a second quadrant angle in standard position whose terminal side punctures the circle (centered at the origin and with radius 13) at the point P(−5,12). State the six trigonometric ratios of θ.

Solution:

The second quadrant triangle with sides −5, 12, and 13 is the right triangle associated with the second quadrant angle θ. Using this triangle with −5 as the adjacent side, 12 as the opposite side, and 13 as the hypotenuse, we have

$$\sin(\theta) = \frac{\text{O}}{\text{H}} = \frac{12}{13} \qquad \cos(\theta) = \frac{\text{A}}{\text{H}} = \frac{-5}{13} = -\frac{5}{13} \qquad \tan(\theta) = \frac{\text{O}}{\text{H}} = \frac{12}{-5} = -\frac{12}{5}$$

$$\csc(\theta) = \frac{13}{12} \qquad \sec(\theta) = -\frac{13}{5} \qquad \cot(\theta) = -\frac{5}{12}$$

Notes: If $\alpha = \theta + 360^\circ$, then α will have the same second quadrant associated triangle as θ and so will have the same trig ratios. This is true for **all** angles of the form $\theta + 360k^\circ$, where $k \in \mathbb{Z}$. Note that of the three ratios sin, cos, and tan, only sin is positive. This is where the S in the CAST RULE (for the **S**in in the second quadrant) comes from.
Another Note: We could have used the circle with radius 1 and the associated triangle with sides $\frac{12}{13}$ and $-\frac{5}{13}$. The resulting triangle is **similar** to the one we used above and so the trig ratios would be the same.

And One More Note: Every angle in standard position will have an "associated" right triangle. Take the end point (a,b) on the terminal arm (the point where θ, in standard position, punctures the circle) and draw a perpendicular to the *x* axis. There is your triangle. The adjacent side is *a*, the opposite is *b*, and the hypotenuse is $\sqrt{a^2+b^2}$.

Common error: Mislabeling the signs on the co-ordinates of the point where θ punctures the circle: for example, mislabeling P as (5,12) instead of (−5,12).

Practice: Suppose θ, where $\pi < \theta < \frac{3\pi}{2}$, punctures the circle, centred at the origin and with radius 5, at the point $(-4,-3)$. State the sin, cos, and tan of θ.
(Hint: hypotenuse = 5)

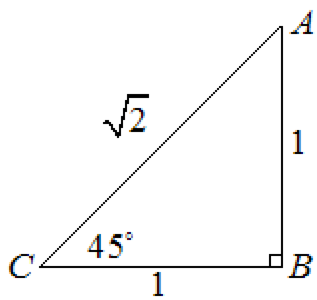
Answer: $\sin(\theta) = -\frac{3}{5}$ $\cos(\theta) = -\frac{4}{5}$ $\tan(\theta) = \frac{3}{4}$

A Great Video for More Details:
https://www.youtube.com/watch?v=V5ArB_GFGYQ
A Great Website for more Detail: https://www.mathsisfun.com/geometry/unit-circle.html

13) Trigonometric Ratios for the 45°, 45°, 90° Triangle

Problem: Find the values of all the six trigonometric ratios of $45^\circ = \dfrac{\pi}{4}$.

Solution: Let $\Delta\, ABC$ be an isosceles right triangle with $\angle\mathrm{B}=90^\circ$ and $AB = CB = 1$. Then $\angle A = \angle C = 45^\circ$ and $AC = \sqrt{2}$.



$\sin(45^\circ) = \dfrac{O}{H} = \dfrac{1}{\sqrt{2}}$ $\quad\cos(45^\circ) = \dfrac{A}{H} = \dfrac{1}{\sqrt{2}}$ $\quad\tan(45^\circ) = \dfrac{O}{A} = 1$

$\csc(45^\circ) = \sqrt{2}$ $\quad\sec(45^\circ) = \sqrt{2}$ $\quad\cot(45^\circ) = 1$

Note: We set $AB = CB = 1$. If we had set $AB = CB = 3,$ then we we would have $AC = 3\sqrt{2}$. Then, $\sin(45^\circ) = \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}},$ which is the same as before. In other words, the sides stay in proportion to one another (similar triangles!) and the ratios are unchanged.

Common error: Determining the hypotenuse $AC = 2$.

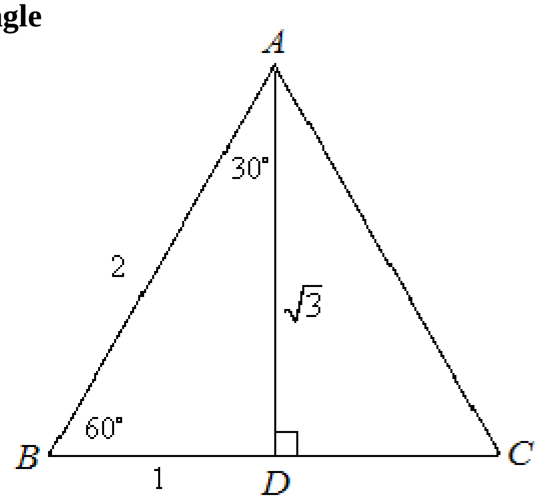
Practice: Find the values of the six trigonometric ratios of the second quadrant angle 135°. (Hint: 135° has related first quadrant angle $180^\circ - 135^\circ = 45^\circ$. Use the ratios for 45° and the CAST RULE.)

Answer: $\sin(135^\circ) = \dfrac{1}{\sqrt{2}}$ $\quad\cos(135^\circ) = -\dfrac{1}{\sqrt{2}}$ $\quad\tan(135^\circ) = -1$

$\csc(135^\circ) = \sqrt{2}$ $\quad\sec(135^\circ) = -\sqrt{2}$ $\quad\cot(135^\circ) = -1$

A Great Video for More Details: https://www.youtube.com/watch?v=hftTj9RfuxM
A Great Website for more Detail:
https://www.purplemath.com/modules/specang.htm

14) Trigonometric Ratios for the 30°, 60°, 90° Triangle



Problem: Find the values of all the six trigonometric ratios of $60^\circ = \frac{\pi}{3}$ and $30^\circ = \frac{\pi}{6}$.

Solution: Let ΔABC be an equilateral triangle with $AB = CB = AC = 2$. Then $\angle A = \angle B = \angle C = 60^\circ$. Drop a perpendicular from A to meet BC at D. $\therefore \Delta ADB \cong \Delta ADC$ and so $\angle BAD = \angle CAD = 30^\circ$ and $BD = CD = 1$.

Finally, using Pythagoras, $AD = \sqrt{2^2 - 1^2} = \sqrt{3}$.

$\sin(60^\circ) = \frac{O}{H} = \frac{\sqrt{3}}{2}$ $\cos(60^\circ) = \frac{A}{H} = \frac{1}{2}$ $\tan(60^\circ) = \frac{O}{A} = \sqrt{3}$

$\csc(60^\circ) = \frac{2}{\sqrt{3}}$ $\sec(60^\circ) = 2$ $\cot(60^\circ) = \frac{1}{\sqrt{3}}$

$\sin(30^\circ) = \frac{O}{H} = \frac{1}{2}$ $\cos(30^\circ) = \frac{A}{H} = \frac{\sqrt{3}}{2}$ $\tan(30^\circ) = \frac{O}{A} = \frac{1}{\sqrt{3}}$

$\csc(30^\circ) = 2$ $\sec(30^\circ) = \frac{2}{\sqrt{3}}$ $\cot(30^\circ) = \sqrt{3}$

Note: The adjacent side for 60°, 1, is the opposite side for 30°. The opposite side for 60°, $\sqrt{3}$, is the adjacent side for 30°. This **always** happens with "complementary" angles. The adjacent for θ is the opposite for $90^\circ - \theta$.

Common error: Forgetting where to put the 1 and the $\sqrt{3}$. Everybody remembers that 2 is the hypotenuse.

Practice: Find the sin, cos and tan of the third quadrant angle 210°. (Hint: 210° has related first quadrant angle $210 - 180 = 30^\circ$. Use the ratios for 30° and the CAST RULE.)

Answer: $\sin(210^\circ) = -\frac{1}{2}$ $\cos(210^\circ) = -\frac{\sqrt{3}}{2}$ $\tan(210^\circ) = \frac{1}{\sqrt{3}}$

A Great Video for More Details: https://www.youtube.com/watch?v=hftTj9RfuxM
A Great Website for more Detail:
https://www.purplemath.com/modules/specang.htm

15) Trigonometric Ratios for the 0°, 90°, 180°, 270°; ie., For Angles ON the Axes

First PLEASE re-read "11) Trigonometric Ratios Using the Circle: Part I".

Problem: Find the six trig ratios for $0^\circ = 0$ radians and $90^\circ = \dfrac{\pi}{2}$ radians.

Solution: $\theta = 0^\circ$ punctures the unit circle at $(1,0)$ and 90° punctures this circle at $(0,1)$.

0°

$$\boxed{\begin{array}{l} \sin(0^\circ)=0 \quad \cos(0^\circ)=1 \quad \tan(0^\circ)=\dfrac{0}{1}=0 \\ \csc(0^\circ) \text{ is undefined} \quad \sec(0^\circ)=1 \quad \cot(0^\circ) \text{ is undefined} \end{array}}$$

90°

$$\boxed{\begin{array}{l} \sin(90^\circ)=1 \quad \cos(90^\circ)=0 \quad \tan(90^\circ) \text{ is undefined} \\ \csc(90^\circ)=1 \quad \sec(90^\circ) \text{ is undefined} \quad \cot(90^\circ)=0 \end{array}}$$

Notes: Division by zero is undefined and that is why there are undefined trig ratios for angles that puncture the axes, that is, where one of the co-ordinates is 0. Also,

the ratios for $360k^\circ$, for $k \in \mathbb{Z}$, are the same as for 0°;

the ratios for $90^\circ + 360k^\circ$, for $k \in \mathbb{Z}$, are the same as for 90°.

Common error: $\sin(0^\circ)=1$, $\cos(0^\circ)=0$, etc.

Practice: Find the sin, cos and tan of (i) $\dfrac{7\pi}{2}$ (ii) -3π

Hint: $\dfrac{7\pi}{2}$ is co-terminal with $-\dfrac{\pi}{2}$ and punctures the circle at $(0,-1)$;

-3π is co-terminal with π and punctures the circle at $(-1,0)$.

Answer: (i) $\sin\left(\dfrac{7\pi}{2}\right)=-1 \quad \cos\left(\dfrac{7\pi}{2}\right)=0 \quad \tan\left(\dfrac{7\pi}{2}\right)$ is undefined

(ii) $\sin(-3\pi)=0 \quad \cos(-3\pi)=-1 \quad \tan(-3\pi)=0$

A Great Video for More Details:
https://www.youtube.com/watch?v=OdmA_ctmcfE
A Great Website for More Detail:
https://www.mathstips.com/trigonometric-ratios-angles-quadrant/

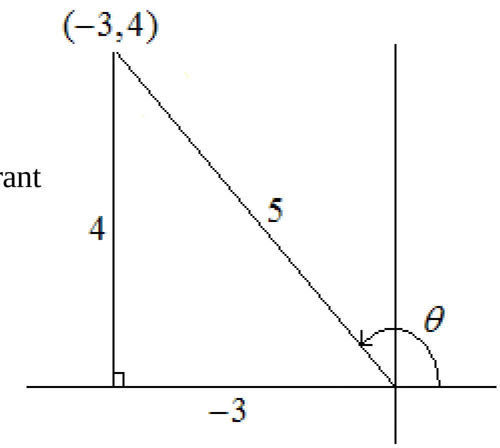
16) CAST RULE

Problems: 1) Given $\tan(\theta) = -4/3$, find the values of $\sin(\theta)$ and $\cos(\theta)$ if

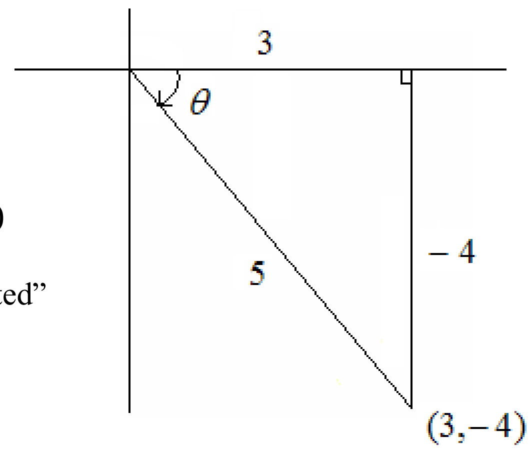
(i) θ is a second quadrant angle.

(ii) θ is a fourth quadrant angle.

2) Why can't θ be a first or third quadrant angle?



Solutions: (i) If $\tan(\theta) = -\frac{4}{3}$, and θ is a second quadrant angle, then $x < 0$ and $y > 0$. Draw θ with terminal point $(-3, 4)$ and drop a perpendicular to the x axis to get the "associated" right triangle. (In the picture, we assumed $\frac{\pi}{2} < \theta < \pi$, but θ could have satisfied, for example, $-\frac{3\pi}{2} < \theta < -\pi$.) Then $\sin(\theta) = \frac{4}{5}$ and $\cos(\theta) = -\frac{3}{5}$.



(ii) If $\tan(\theta) = -\frac{4}{3}$, and θ is a fourth quadrant angle, then $x > 0$ and $y < 0$. Draw θ with terminal point $(3, -4)$ and drop a perpendicular to the x axis to get the "associated" right triangle. Then $\sin(\theta) = -\frac{4}{5}$ and $\cos(\theta) = \frac{3}{5}$.

2) tan is **positive** in quadrants 1 and 3 so we can't have $\tan(\theta) = -\frac{4}{3}$!

Note:

QUADRANT 4	QUADRANT 1	QUADRANT 2	QUADRANT 3
C	A	S	T
COS	ALL	SIN	TAN

Common error: Given $\tan(\theta) = -\frac{4}{3}$ in the second quadrant, setting $x = 3$ and $y = -4$.

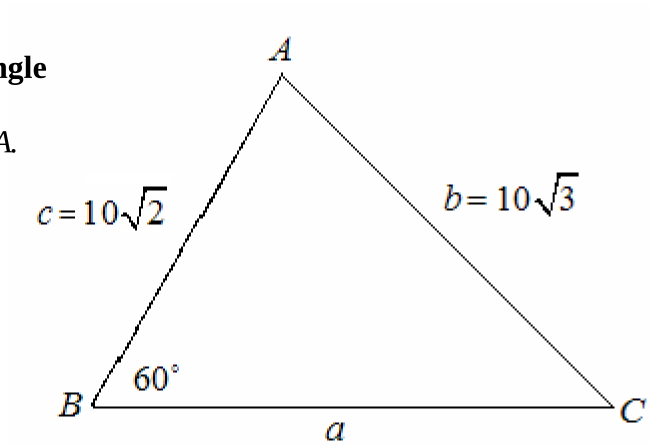
Practice: Given $\cos(\theta) = -\frac{2}{3}$ and θ is a third quadrant angle, find $\sin(\theta)$ and $\tan(\theta)$.

Answer: $\sin(\theta) = -\frac{\sqrt{5}}{3}$ $\quad \tan(\theta) = \frac{\sqrt{5}}{2}$

A Great Video for More Details:
https://www.youtube.com/watch?v=SszYUNLH80Q
A Great Website for More Detail:
https://www.onlinemathlearning.com/trig-equations-cast.html

17) Sine Law: Find an Angle

Problem: Find $\angle C$ and $\angle A$.



Solution: By the Sine Law,

$$\frac{\sin(C)}{c} = \frac{\sin(B)}{b}$$

$$\therefore \sin(C) = \frac{c}{b}\sin(B) \overset{\boxed{b=10\sqrt{3},\ c=10\sqrt{2},\ \sin(60^\circ)=\frac{\sqrt{3}}{2}}}{=} \frac{10\sqrt{2}}{10\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$$

$$\therefore \angle C \overset{\boxed{\text{We know the trig ratios for } 45^\circ.}}{=} 45^\circ \text{ and } \angle A = 180^\circ - (60^\circ + 45^\circ) = 75^\circ$$

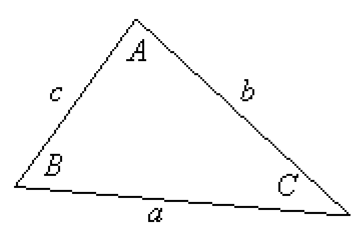
Notes: $\sin(C) = \frac{1}{\sqrt{2}} \Rightarrow C = 45^\circ$ or 135°. (Sin is + in both the first and second quadrants!)

The angles in ΔABC must sum to 180° and so $C = 135^\circ$ is too big. However, sometimes, there are two solutions!

Another Note: This works when you have two sides and an angle which is **not the contained angle**. If you have the contained angle, you need the Cosine Law.

Common error: $a\sin(A) = b\sin(B)$

Practice: In ΔABC, $c = AB = 10$, $b = AC = 12$ and $\angle B = 60^\circ$. Find $\angle C$ and $\angle A$ accurate to one decimal place. (Hint: to solve $\sin(C) = x$ on most calculators, use "x 2nd Function sin =".)



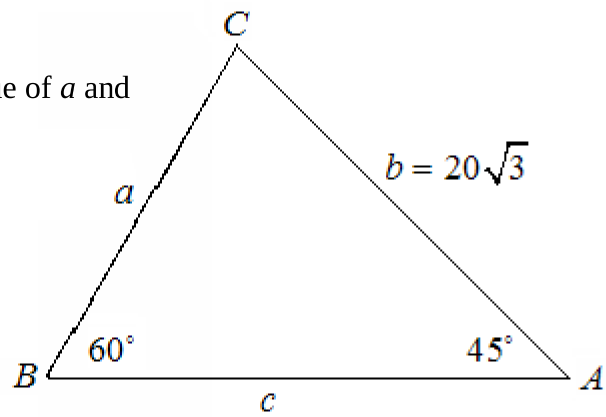
(not drawn to scale!)

Answers: $\angle C \doteq 46.2°$ $\quad \angle A \doteq 73.8°$

A Great Video for More Details: **https://www.youtube.com/watch?v=RCyjglaJo5w**
A Great Website for More Detail:
https://www.mathsisfun.com/algebra/trig-sine-law.html

18) Sine Law: Find a Side

Problem: Find (i) the exact value of a and (ii) c accurate to two decimals.



Solution: $\angle C = 180^\circ - (60^\circ + 45^\circ) = 75^\circ$

By the Sine Law:

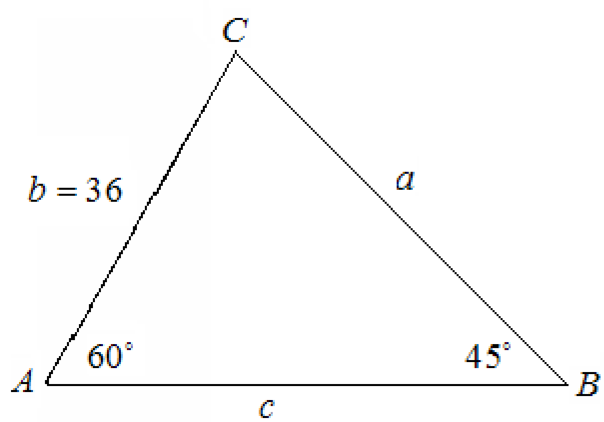
$$a = \sin(A) \times \frac{b}{\sin B} = \sin(45^\circ) \times \frac{20\sqrt{3} \times 2}{\sqrt{3}} = \frac{1}{\sqrt{2}} \times 40 = 20\sqrt{2}$$

$$c = \sin(C) \times \frac{b}{\sin(B)} = \sin(75^\circ) \times \frac{20\sqrt{3} \times 2}{\sqrt{3}} \doteq 0.9659 \times 40 \doteq 38.64$$

Note: The Sine Law contains four quantities: two angles in a triangle and the two sides opposite these angles. The Sine Law is useful when you have three of these four quantities.

Common error: $a\sin(A) = b\sin(B)$

Practice: In ΔABC, $\angle A = 60^\circ$, $\angle B = 45^\circ$, and side $b = AC = 36$. Find the lengths of sides a = BC and c = AB accurate to two decimal places.

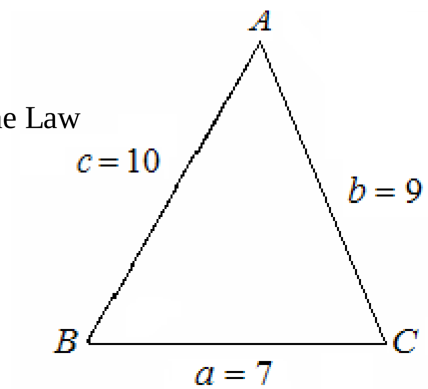


Answer: $a \doteq 44.10$ $\quad c \doteq 49.18$

A Great Video for More Details: https://www.youtube.com/watch?v=9fS0uA4iLxI
A Great Website for More Detail:
https://www.mathsisfun.com/algebra/trig-sine-law.html

19) Cosine Law: Find an angle

Problem: In ΔABC, use the Cosine Law to find $\angle B$ to the nearest degree.



Solution: Using the Cosine Law:

$$\cos(B) = \frac{a^2 + c^2 - b^2}{2ac} = \frac{49 + 100 - 81}{2(7)(10)} = \frac{68}{140} = \frac{17}{35}$$

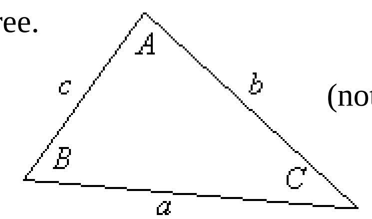
Make sure your calculator is in "DEGREE" mode!

$$\therefore B \underset{}{\overset{\boxed{\substack{\text{On most calculators:}\\ 17/35\ =\ 2\text{ND FUNCTION } \cos\ =}}}{\doteq}} 60.94^\circ$$

Note: Make sure your calculator is in degree mode. To find an angle using the Cosine Law you need to know the lengths of all the three sides of the triangle.

Common error: Using "RADIAN" mode when the answer is required in degrees or vice-versa. The answer here in radians is 1.06. Remember that 1 radian is about 57°. So, 1.06 radians is about 61°.

Practice: In ΔABC, we have a = BC = 2, b = AC = 3 and c = AB = 4. Find $\angle C$ to the nearest degree.



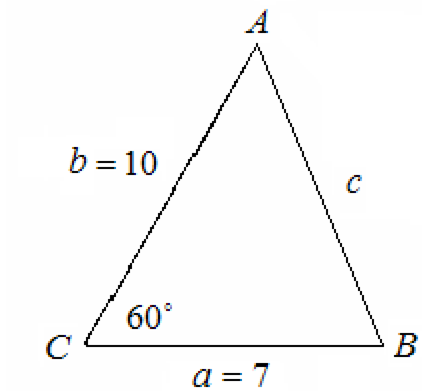
(not drawn to scale!)

Answer: $\angle C = 105^\circ$.

A Geat Video for More Details: https://www.youtube.com/watch?v=9CGY0s-uCUE
A Great Website for More Detail: https://www.mathsisfun.com/algebra/trig-cosine-law.html

20) Cosine Law: Find a Side

Problem: In ΔABC, use the Cosine Law to find $c = AB$ correct to two decimal places.



Solution: Using the Cosine Law:

$$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot cos(C) \overset{\boxed{a=7\ \ b=10\ \ C=60^\circ}}{=} 49 + 100 - 140(0.5) = 79$$

$\therefore\ c \doteq 8.89$

Note: To find a side using the Cosine Law, you need two sides and the **contained** angle. If you have a "non-contained" angle, use the Sine Law to find the contained angle and then use the Cosine Law or the Sine Law to find the required side.

Common error: $c^2 = a^2 + b^2 + 2ab\cos(C)$

Practice: In ΔABC, we have b = AC = 6, c = AB = 4 and $\angle A = 35^\circ$. Find a = BC, correct to two decimals.

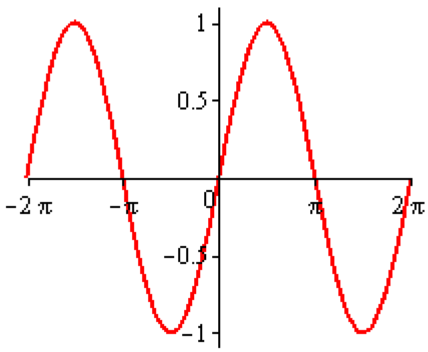
Answer: $a \doteq 3.56$

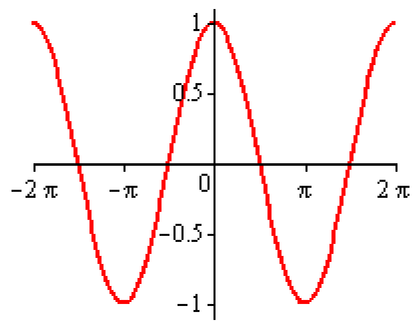
A Great Video for More Details: **https://www.youtube.com/watch?v=9CGY0s-uCUE**
A Great Website for More Detail: **https://www.mathsisfun.com/algebra/trig-cosine-law.html**

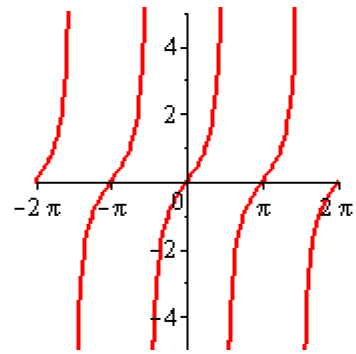
21) The Graphs of the Sin, Cos, and Tan Functions

Problem: Graph for $0 \le \theta \le 2\pi$: (i) $y = \sin(\theta)$ (ii) $y = \cos(\theta)$ (iii) $y = \tan(\theta)$

Solution: (i) $y = \sin(\theta)$ (ii) $y = \cos(\theta)$ (iii) $y = \tan(x)$

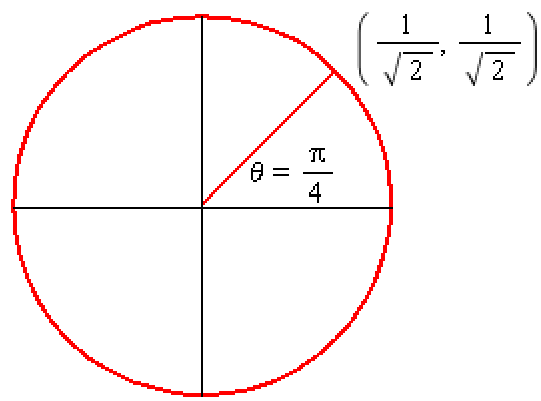






Note: The range of $y = \sin(\theta)$ and $y = \cos(\theta)$ is $[-1,1]$. The range of $y = \tan(\theta)$ is $\mathbb{R}$.

Common error: Confusing, for example, the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, which is on the graph of both $y = \sin(\theta)$ and $y = \cos(\theta)$, with the terminal point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ of $\theta = \frac{\pi}{4}$, on the circle $x^2 + y^2 = 1$. On this circle, for each angle θ, we puncture the circle in a point (x, y) and we define $x = \cos(\theta)$ and $y = \sin(\theta)$. The values of cos and sin lead to the graphs above.



Practice: Name all values of θ where (i) $\sin(\theta) = 0$ (ii) $\cos(\theta) = 0$ (iii) $\tan(\theta) = 0$

Answer: (i) $k\pi,\ k \in \mathbb{Z}$ (ii) $\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$ (iii) $k\pi,\ k \in \mathbb{Z}$

A Great Video for More Details: https://www.youtube.com/watch?v=-nr5KCitdGw
A Great Website for More Detail: https://www.mathsisfun.com/algebra/trig-sin-cos-tan-graphs.html

22) Period of the Sin, Cos, and Tan Functions

Problems: State the period of each of the following:

1) (i) $y=\sin(x)$ (ii) $y=\sin(2x)$ (iii) $y=\sin\left(\frac{x}{2}\right)$

2) (i) $y=\cos(x)$ (ii) $y=\cos(3x)$ (iii) $y=\cos\left(\frac{x}{3}\right)$

3) (i) $y=\tan(x)$ (ii) $y=\tan(4x)$ (iii) $y=\tan\left(\frac{\pi}{2}x\right)$

Solutions: 1)(i) 2π (ii) $\frac{2\pi}{2}=\pi$ (iii) $\frac{2\pi}{\left(\frac{1}{2}\right)}=4\pi$

2)(i) 2π (ii) $\frac{2\pi}{3}$ (iii) $\frac{2\pi}{\left(\frac{1}{3}\right)}=6\pi$

3)(i) π (ii) $\frac{\pi}{4}$ (iii) $\frac{\pi}{\left(\frac{\pi}{2}\right)}=2$

Note: If the function $y=f(x)$ has period P, then $y=f(kx)$ has period $\frac{P}{k}$.

Common error: The period of $y=\tan(2x)$ is $\frac{2\pi}{2}=\pi$.

Practice: Because the period of $y=\sin(x)$ is 2π, we know that for any integer k, $\sin(x+2k\pi)=$ _____

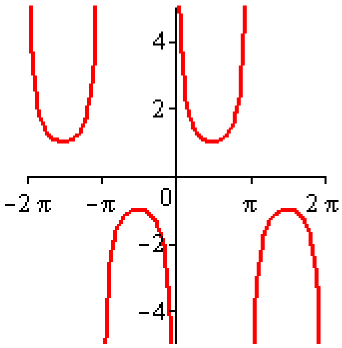
Answer: $\sin(x)$

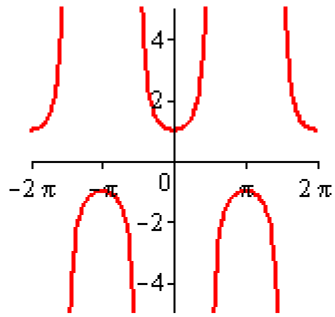
A Great Video for More Details: https://www.youtube.com/watch?v=flF2BYFLdkU
A Great Website for More Detail:
https://www.purplemath.com/modules/grphtrig.htm

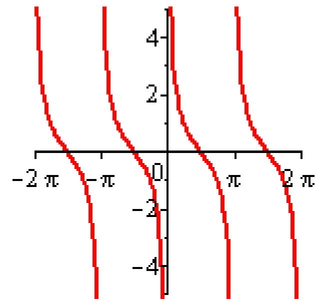
23) The Graphs of the Csc, Sec, and Cot Functions

Problem: Graph for $0 \le \theta \le 2\pi$: (i) $y = \csc(\theta)$ (ii) $y = \sec(\theta)$ (iii) $y = \cot(\theta)$

Solution: (i) $y = \csc(\theta)$ (ii) $y = \sec(\theta)$ (iii) $y = \cot(\theta)$







Note: The range of $y = \csc(\theta)$ and $y = \sec(\theta)$ is $(-\infty, -1] \cup [1, \infty)$. The range of $y = \cot(\theta)$ is $\mathbb{R}$.

Common error: Completely misunderstanding why the range of csc is from +1 up and from -1 down: when $\sin(\theta)$ is inside $(-1,1)$, then $\frac{1}{\sin(\theta)} = \csc(\theta)$ is outside $(-1,1)$. When $\sin(\theta) = \pm 1$, so does $\csc(\theta)$.

Practice: Name all values of θ where (i) $\csc(\theta)$ (ii) $\sec(\theta)$ (iii) $\cot(\theta)$ is undefined.

Answer: (i) $k\pi,\ k \in \mathbb{Z}$ (ii) $\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$ (iii) $k\pi,\ k \in \mathbb{Z}$

A Great Video for More Details: https://calcworkshop.com/graphing-trig-functions/graphing-reciprocal-trig-functions/
A Great Website for More Detail:
https://calcworkshop.com/graphing-trig-functions/graphing-reciprocal-trig-functions/ (Yes, the same as the video link!)

24) Trig Formulas That You Should Know

Problem: Complete the following formulas:

(i) $\sin^2(\theta)+\cos^2(\theta)=\boxed{}$; $1+\tan^2(\theta)=\boxed{}$

(ii) $\sin(-\theta)=\boxed{}$; $\cos(-\theta)=\boxed{}$

(iii) $\sin\left(\frac{\pi}{2}-\theta\right)=\boxed{}$; $\cos\left(\frac{\pi}{2}-\theta\right)=\boxed{}$

(iv) $\sin(A\pm B)=\boxed{}$; $\cos(A\pm B)=\boxed{}$

(v) $\sin(2A)=\boxed{}$; $\cos(2A)=\boxed{}$

Solution: (i) $\sin^2(\theta)+\cos^2(\theta)=\boxed{1}$; $1+\tan^2(\theta)=\boxed{\sec^2(\theta)}$

(ii) $\sin(-\theta)=\boxed{-\sin(\theta)}$; $\cos(-\theta)=\boxed{\cos(\theta)}$

(iii) $\sin\left(\frac{\pi}{2}-\theta\right)=\boxed{\cos(\theta)}$; $\cos\left(\frac{\pi}{2}-\theta\right)=\boxed{\sin(\theta)}$

(iv) $\sin(A\pm B)=\boxed{\sin(A)\cos(B)\pm\cos(A)\sin(B)}$;

(v) $\cos(A\pm B)=\boxed{\cos(A)\cos(B)\mp\sin(A)\sin(B)}$

(vi) $\sin(2A)=\boxed{2\sin(A)\cos(A)}$; $\cos(2A)=\boxed{\cos^2(A)-\sin^2(A)}$

Note: These formulas are used over and over and over again. Learn them.

Common error: $\cos(A\pm B)=\boxed{\cos(A)\cos(B)\pm\sin(A)\sin(B)}$

Practice: Complete the formulas:

(i) $\cot^2(\theta)+1=\boxed{}$ (ii) $\tan(-\theta)=\boxed{}$ (iii) $\tan\left(\frac{\pi}{2}-\theta\right)=\boxed{}$

(iv) $\tan(A\pm B)=\boxed{}$ (v) $\tan(2A)=\boxed{}$

Answer: (i) $\boxed{\csc^2(\theta)}$ (ii) $\boxed{-\tan(\theta)}$ (iii) $\tan\left(\frac{\pi}{2}-\theta\right)=\boxed{\cot(\theta)}$

(iv) $\tan(A+B)=\boxed{\dfrac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}}$ (v) $\tan(2A)=\boxed{\dfrac{2\tan(A)}{1-\tan^2(A)}}$

A Great Video for More Details: https://www.youtube.com/watch?v=a70-dYvDJZY
A Great Website for More Detail:
https://www.purplemath.com/modules/idents.htm

Solutions Part 9
LOGS

AND
MATH EXPONENTS!

1) Exponents

Problems: 1) Evaluate: (i) 2^3 (ii) $\left(\frac{3}{5}\right)^3$ (iii) 4^{-2} (iv) 10^0 (v) $\left(\frac{1}{0.01}\right)^{-3}$

2) Simplify: (i) $\frac{x^5x^4}{x^7}$ (ii) w^{-1} (iii) $\frac{1}{w^{-3}}$ (iv) $\left(z^{\frac{2}{3}}\right)^{10}$ (v) $\left(\frac{a^7b^3}{c^2}\right)^5$

Solution: 1) (i) $2^3 = 2\times2\times2 = 8$ (ii) $\left(\frac{3}{5}\right)^3 = \frac{3^3}{5^3} = \frac{27}{125}$ (iii) $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

(iv) $10^0 = 1$ (v) $\left(\frac{1}{0.01}\right)^{-3} = (0.01)^3 = (10^{-2})^3 = 10^{-6} = \frac{1}{1000000}$

2) (i) $\frac{x^5x^4}{x^7} = x^{5+4-7} = x^2$ (ii) $w^{-1} = \frac{1}{w}$ (iii) $\frac{1}{w^{-3}} = w^3$ (iv) $\left(z^{2/3}\right)^{10} = z^{20/3}$ (v) $\left(\frac{a^7b^3}{c^2}\right)^5 = \frac{a^{35}b^{15}}{c^{10}}$

Notes: Multiplication is a short form for repeated addition: $7+7+7+7 = 7\times4$
Exponentiation is a short form for repeated multiplication: $7\times7\times7\times7 = 7^4$

Common error: $\left(3^5\right)^4 = 3^9$, ie., adding exponents when you should be multiplying.

Practice: 1) Evaluate: (i) 2^5 (ii) $\left(\frac{2}{3}\right)^{-2}$ (iii) 10^0 (iv) 0^0

2) Simplify: (i) $\frac{x^5y^2}{x^{11}y^{-5}}$ (ii) h^{-1} (iii) $\frac{1}{h^{-1}}$ (iv) $\left(\frac{ab^{-2}}{c^{-3}}\right)^5$

Answers: 1) (i) 32 (ii) $\frac{9}{4}$ (iii) 1 (iv) Does not exist. "0^0" is undefined.

2) (i) $\frac{y^7}{x^6}$ (ii) $\frac{1}{h}$ (iii) h (iv) $\frac{a^5c^{15}}{b^{10}}$

A Great Video for More Detail: https://www.youtube.com/watch?v=A1wKTiBTsfk
A Great Website for More Detail: purplemath.com/modules/exponent.htm

2) Logarithms (Log means FIND THE EXPONENT!)

Problems: 1) Evaluate: (i) $\log_2 8$ (ii) $\log_2\left(\frac{1}{8}\right)$ (iii) $\log_3 1$ (iv) $\log_5 5$

(v) $\log 10000$ (vi) $\ln(e^7)$*

2) Expand using log properties: $\ln\left(\frac{x^3 y^{1/2}}{z^4}\right)$

3) Change $\log_5 7$ to log with base 3, then with base 10, and finally with base *e*.

> *"e" and "ln" refer to the "natural logarithm". If you have not taken calculus, you may be totally unfamiliar with e. If so, treat it as a constant just as you would, for example, the letter a.*

Solutions: 1) (i) $\log_2 8 = 3$ (ii) $\log_2\left(\frac{1}{8}\right) = -3$ (iii) $\log_3 1=0$ (iv) $\log_5 5 = 1$

(v) $\log 10000 = 4$ (vi) $\ln(e^7) = 7$

2) $\ln\left(\frac{x^3 y^{1/2}}{z^4}\right) = \ln(x^3) + \ln\left(y^{1/2}\right) - \ln\left(z^4\right) = 3\ln x + \frac{1}{2}\ln y - 4\ln z$

3) $\log_5 7 = \frac{\log_3 7}{\log_3 5} = \frac{\log 7}{\log 5} = \frac{\ln 7}{\ln 5}$

Notes: "Log" means "Find the exponent!".
"log" with no base is short for $\log_{10}$ and $\ln$ is short for $\log_e$.

Common error: Many students confuse $\left(\log_2 8\right)^3 = \log_2 8 \times \log_2 8 \times \log_2 8 = 3^3 = 27$ with $\log_2\left(8^3\right) = \log_2\left(8\times 8\times 8\right) = \log_2(2^9) = 9.$

Practice: 1)(i) $\log_3 81$ (ii) $\log_5\left(\frac{1}{125}\right)$ (iii) $\log_{0.1} 1$ (iv) $\log_2 0$

(v) $\log \frac{1}{10000}$ (vi) $\ln\left(\frac{1}{e^7}\right)$

2) Expand using log properties: $\ln\left(x^3 + y^{1/2} - z^4\right)$

3) Change $\log_5 7$ to log with base 7.

Answers: 1) Evaluate: (i) 4 (ii) -3 (iii) 0 (iv) does not exist (v) -4 (vi) -7

2) You can't expand this at all. 3) $\frac{1}{\log_7 5}$.

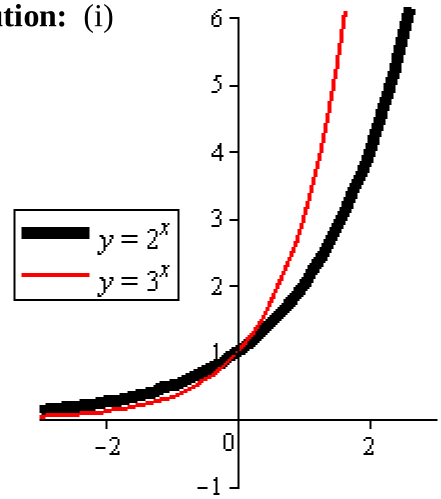
A Great Video for More Detail: https://www.youtube.com/watch?v=i0yb8CjIyWs
A Great Website for More Detail: purplemath.com/modules/logrules.htm

3) Exponential Graphs

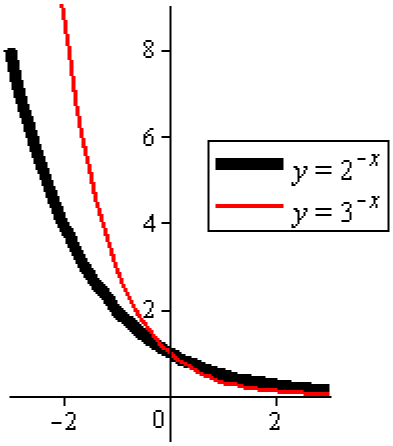
Problem: (i) Graph the exponential functions $y = 2^x$ and $y = 3^x$ on the same set of axes.

(ii) Graph the exponential functions $y = 2^{-x} = \dfrac{1}{2^x}$ and $y = 3^{-x} = \dfrac{1}{3^x}$ on the same set of axes.

Solution: (i)



(ii)

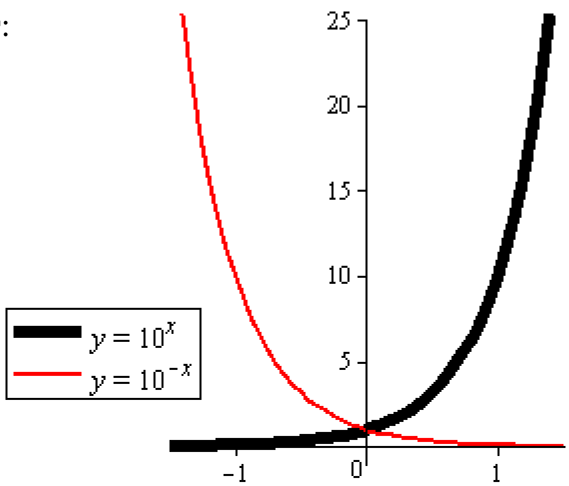


Notes: Because the base is bigger (and greater than 1), $y = 3^x$ goes up faster than $y = 2^x$ when $x > 0$ and y approaches 0 faster when $x \to -\infty$. Also, $y = a^x$ is **always** $+$!
$y = a^{-x}$ and $y = a^x$ are mirror images in the y axis.

Common error: Drawing the graph of $y = 2^x$ so that it appears to cross the x axis when $x \to -\infty$.

Practice: Graph $y = 10^x$ and $y = 10^{-x}$ on the same set of axes. .

Answer:

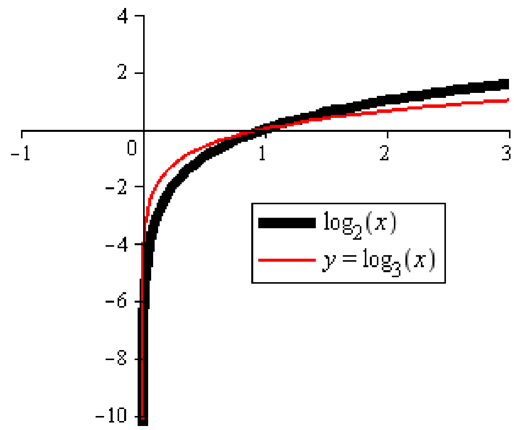


A Great Video for More Detail: https://www.youtube.com/watch?v=DASfP8KAyvs
A Great Website for More Detail: purplemath.com/modules/graphexp.htm

4) Logarithmic Graphs

Problem: Graph the functions $y = \log_2(x)$ and $y = \log_3(x)$ on the same set of axes.

Solution:



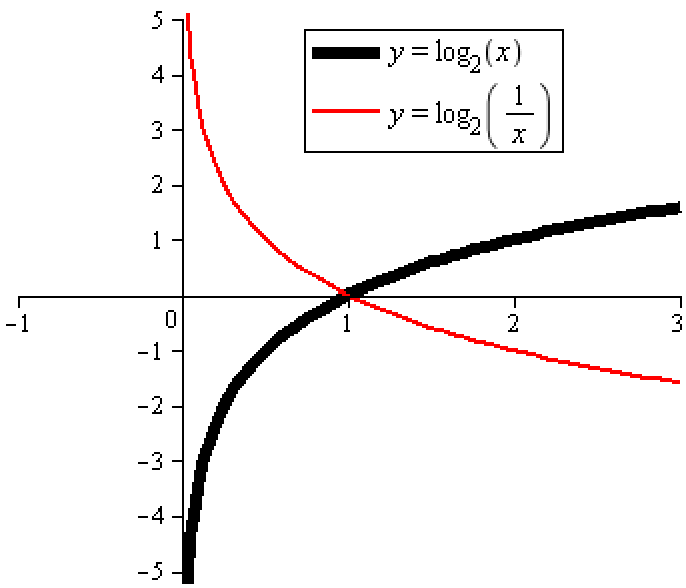
Notes: Because the base is bigger (and greater than 1), $y = \log_3(x)$ goes up more slowly than $y = \log_2(x)$ when $x > 1$ and y approaches $-\infty$ faster when $x \to 0^+$. The domain of $y = \log_a(x)$ is $(0,\infty)$!

Common error: Drawing the graph of $y = \log_2(x)$ so that it appears to cross the y axis when $x \to 0^+$. Even worse, this means you are allowing $x < 0$ into $y = \log_2(x)$. NO!

Practice: Graph $y = \log_2(x)$ and $y = \log_2\left(\frac{1}{x}\right)$ on the same set of axes.

Hint: $\log_2\left(\frac{1}{x}\right) = \log_2\left(x^{-1}\right) = -\log_2(x)$

Answer:



A Great Video for More Detail: https://www.youtube.com/watch?v=2Vvs9hU3pBI
A Great Website for More Detail: purplemath.com/modules/graphlog.htm

5) Exponents to Logarithms and Vice-Versa

Problems: 1) Change to a log equation: (i) 32=2^5 (ii) $y=10^x$ (iii) $y=4x^k$ (Use base 10.)
2) Change to an exponential equation: (i) $\log_3 81=4$ (ii) $y=\log_5 x$

Solutions: 1)(i) $32=2^5 \Leftrightarrow 5=\log_2 32$ (ii) $y=10^x \Leftrightarrow x=\log_{10}(y)=\log(y)$
(iii) $y=4x^k \overset{\boxed{\text{Take the log of each side}}}{\Leftrightarrow} \log(y)=\log(4x^k) \overset{\boxed{\text{log properties}}}{=} \log(4)+k\log(x)$
2)(i) $\log_3 81=4 \Leftrightarrow 81=3^4$ (ii) $y=\log_5 x \Leftrightarrow x=5^y$

Note: $y=a^x \Leftrightarrow x=\log_a(y)$

Common error: $y=4x^k \Leftrightarrow k=\log_{4x}(y)$

Practice: 1) Change to a log equation: (i) $\frac{8}{27}=\left(\frac{2}{3}\right)^3$ (ii) $y=0.5x^{k-1}$ (Use base 10.)
2) Change to an exponential equation: (i) $\log(0.001)=-3$ (ii) $y+2=\log_5(3x-1)$

Answers: 1)(i) $\log_{2/3}\left(\frac{8}{27}\right)=3$ (ii) $\log(y)=\log(.5)+(k-1)\log(x)$
2)(i) $10^{-3}=.001$ (ii) $5^{y+2}=3x-1$

A Great Video for More Detail:
https://www.youtube.com/watch?v=cnxiWK_KY6U
A Great Website for More Detail: **purplemath.com/modules/logs.htm**

6) Using a Calculator to Evaluate Exponents and Logs

Problems: Use a calculator to give answers rounded to two decimal places.

1) (i) $2^{3.3}$ (ii) $5^{1/7}$ (iii) $(-10)^{1/3}$

2) (i) $\log(25)$ (ii) $\ln(25)*$ (iii) $\log_2(25)$

> *"e" and "ln" refer to the "natural logarithm". If you have not taken calculus, you may be totally unfamiliar with e. If so, treat it as a constant just as you would, for example, the letter a.*

Solutions: 1)(i) $2^{3.3} \overset{\boxed{\text{Most Calculators: } 2.2\ y^x\, 3.3=}}{\doteq} 9.85$ (ii) $\sqrt[7]{5}=5^{1/7} \overset{\boxed{\text{Most Calculators: } 7\ \sqrt[x]{y}\ 5=}}{\doteq} 1.26$

(iii) $(-10)^{1/3}$. Most calculators won't accept a base < 0.

We know the answer should be negative. So

$(-10)^{1/3} = -(10)^{1/3} \overset{\boxed{\text{Most Calculators: } 3\ \sqrt[y]{x}\ 10=}}{\doteq} -2.15$

2)(i) $\log(25) \overset{\boxed{\text{Most Calculators: } 25\ \log=}}{\doteq} 1.40$ (ii) $\ln(25) \overset{\boxed{\text{Most Calculators: } 25\ \ln=}}{\doteq} 3.22$

(iii) $\log_2(25) \overset{\boxed{\text{Most Calculators: } 25\ \log/2\log=}}{\doteq} 4.64$ OR $\overset{\boxed{\text{Most Calculators: } 25\ \ln/2\ln=}}{\doteq} 4.64$

Note: Use the change of base formula when given a base other than 10 or *e*:

$$\log_a(b) = \frac{\log(b)}{\log(a)} = \frac{\ln(b)}{\ln(a)}$$

Common error: Believing your calculator when it answers INCORRECTLY when you enter $-2\ y^x\ 3$. Most calculators will report, "error" because they are not programmed to handle a negative base. Of course, $(-2)^3 = -8$.

Practice: Use a calculator to give answers rounded to two decimal places.

1)(i) $5^{3.1}$ (iii) $2^{1/5}$ (iii) $(-1001)^{1/3}$

2)(i) $\log(6)$ (ii) $\ln(6)$ (iii) $\log_7(6)$

Answers: 1)(i) 146.83 (ii) 1.15 (iii) −10.00

2)(i) .78 (ii) 1.79 (iii) .92

A Great Video for More Detail:
https://www.youtube.com/watch?v=OkFdDqW9xxM
A Great Website for More Detail:
https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:logs/x2ec2f6f830c9fb89:change-of-base/a/logarithm-change-of-base-rule-intro

Solutions Part 10
CALCULUS

$$\boxed{\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Differentiation

$\frac{d}{dx}(\text{constant}) = 0$

$\frac{d}{dx} x^n = nx^{n-1}$

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} \ln(x) = \frac{1}{x}$

$\frac{d}{dx} \sin(x) = \cos(x)$

$\frac{d}{dx} \cos(x) = -\sin(x)$

$\frac{d}{dx} \tan(x) = \sec^2(x)$

$\frac{d}{dx} \csc(x) = -\csc(x)\cot(x)$

$\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$

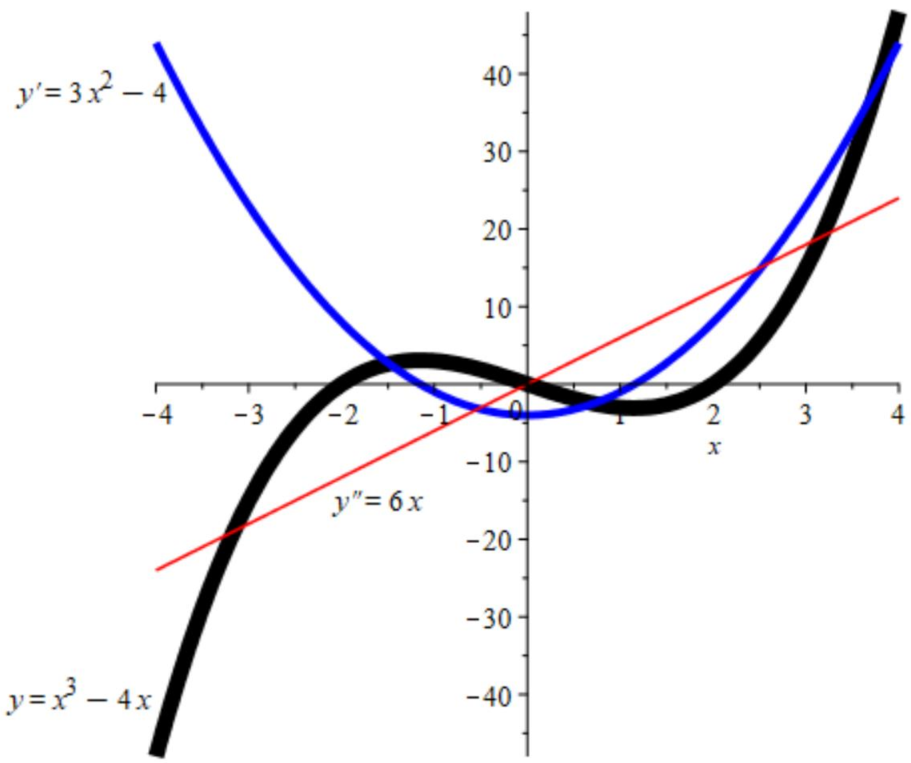
$\frac{d}{dx} \cot(x) = -\csc^2(x)$

$(f \pm g)' = f' \pm g'$

k constant $(kf)' = kf'$

$(fg)' = f'g + fg'$

$(f/g)' = (gf' - fg')/g^2$



1) Limits

Problem: Evaluate the following limits:

(i) $\lim_{x\to 3}\frac{x^2-9}{x^3-27}$ (ii) $\lim_{x\to -4^+}\frac{x^2+5}{x^2+x-12}$ (iii) $\lim_{\theta\to 0}\frac{2\theta}{\sin(3\theta)}$ (iv) $\lim_{x\to\infty}\frac{3x+5}{4x-7}$

Solution: (i) $\lim_{x\to 3}\frac{x^2-9}{x^3-27}=\lim_{x\to 3}\frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x^2+3x+9)}=\lim_{x\to 3}\frac{(x+3)}{(x^2+3x+9)}=\frac{6}{27}=\frac{2}{9}$

(ii) $\lim_{x\to -4^+}\frac{x^2+5}{x^2+x-12}=\lim_{x\to -4^+}\frac{x^2+5}{(x+4)(x-3)}=-\frac{21}{7}\lim_{x\to -4^+}\frac{1}{x+4}=-\infty$

(iii) $\lim_{\theta\to 0}\frac{2\theta}{\sin(3\theta)}=\lim_{\theta\to 0}\left(\frac{3\theta}{\sin(3\theta)}\left(\frac{2}{3}\right)\right)=\frac{2}{3}\lim_{\theta\to 0}\frac{3\theta}{\sin(3\theta)}=\frac{2}{3}\cdot 1=\frac{2}{3}$

(iv) $\lim_{x\to\infty}\frac{3x+5}{4x-7}=\lim_{x\to\infty}\frac{\left(\frac{3x+5}{x}\right)}{\left(\frac{4x-7}{x}\right)}=\lim_{x\to\infty}\left(\frac{3+\frac{5}{x}}{4-\frac{7}{x}}\right)=\frac{3}{4}$

Note: $\lim_{x\to -4^+}\frac{1}{x+4}=\infty$ $\lim_{x\to -4^-}\frac{1}{x+4}=-\infty$

Common error: "$\lim_{x\to 3}\frac{x^2-9}{x^3-27}=\frac{0}{0}=0$" or "$\lim_{x\to 3}\frac{x^2-9}{x^3-27}=\frac{0}{0}=\infty$"

Practice:

(i) $\lim_{x\to -5}\frac{x^3+125}{x^2+6x+5}$ (ii) $\lim_{x\to -4^-}\frac{x^2+5}{x^2+x-12}$ (iii) $\lim_{\theta\to 0}\frac{\theta\cos(\theta)}{\sin(\theta)}$ (iv) $\lim_{x\to -\infty}\frac{12x^2-x}{4x+7x^3}$

Answer: (i) $-\frac{75}{4}$ (ii) ∞ (iii) 1 (iv) 0

A Great Video for More Details:
https://www.youtube.com/watch?v=QfPqRMqP5kU
A Great Website for More Details: https://www.cliffsnotes.com/study-guides/calculus/calculus/limits/infinite-limits

2) The Derivative from First Principles (that is, from the Definition of the Derivative)

Problem: Find the derivative $\frac{dy}{dx}$ from first principles, where $y = 2x^2 - 3x + 1$.

Solution: $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{2(x+h)^2 - 3(x+h) + 1 - (2x^2 - 3x + 1)}{h}$$
$$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 3x - 3h + 1 - 2x^2 + 3x - 1}{h}$$
$$= \lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h}$$
$$= \lim_{h \to 0} \frac{h(4x + 2h - 3)}{h}$$
$$= \lim_{h \to 0} \left(4x + 2h - 3\right)$$
$$= 4x - 3$$

Note: Another approach is to find $\frac{dy}{dx}_{|x=x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

Common error: $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2x^2 + h - 3x + h + 1 - (2x^2 - 3x + 1)}{h}$

Practice: Given that $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h \to 0} \frac{1 - \cos(h)}{h} = 0,$ use these two limits to find $\frac{d(\sin(x))}{dx}$ from first principles.

Answer: $\frac{d(\sin(x))}{dx} = \cos(x)$

A Great Video for More Details:
https://www.youtube.com/watch?v=IWpsnR2uRus
A Great Website for More Detail: **https://www.intmath.com/differentiation/3-derivative-first-principles.php**

3) Using the Derivative Rules

Problem: Use the derivative rules to find $\frac{dy}{dx}$, $\frac{ds}{dt}$, and $\frac{dw}{dz}$. Do not simplify your answers. (i) $y=(\pi x+3x^4)\sin(x)$ (ii) $s=\frac{\cos(t)-t}{e^t+et}$ (iii) $w=\left(\sin(z^{\frac{1}{3}})+z^{-\frac{1}{3}}\right)^5$

Solution: (i) Using the product rule, $\frac{dy}{dx}=(\pi x+3x^4)\cos(x)+\sin(x)(\pi+12x^3)$

(ii) Using the quotient rule, $\frac{ds}{dt}=\frac{\left(e^t+et\right)(-\sin(t)-1)-(\cos(t)-t)\left(e^t+e\right)}{\left(e^t+e\right)^2}$

(iii) Using the chain rule, $\frac{dw}{dz}=5\left(\sin(z^{\frac{1}{3}})+z^{-\frac{1}{3}}\right)^4\left(\cos(z^{\frac{1}{3}})\left(\frac{1}{3}z^{-\frac{2}{3}}\right)-\frac{1}{3}z^{-\frac{4}{3}}\right)$

Note: Using the chain rule in (iii), we first took the derivative of the power function with respect to the inside $\left(\sin(z^{\frac{1}{3}})+z^{-\frac{1}{3}}\right)$, then the derivative of sin with respect to the inside $z^{\frac{1}{3}}$, and, finally, the derivative of $z^{\frac{1}{3}}$ and $z^{-\frac{1}{3}}$ with respect to z.

Common error: $\frac{dw}{dz}=5\left(\cos(z^{\frac{1}{3}})-\frac{1}{3}z^{-\frac{4}{3}}\right)^4$

Practice: Find the derivatives $\frac{dy}{dx}$, $\frac{ds}{dt}$, and $\frac{dw}{dz}$. Do not simplify your answers.

(i) $y=x^2\cos(x)$ (ii) $s=\frac{t^4}{\sin(t)+1}$ (iii) $w=\left(e^{2z}-(3z+1)^2\right)^{-\frac{1}{4}}$

Answer:

(i) $\frac{dy}{dx}=x^2(-\sin(x))+\left(\cos(x)\right)(2x)$

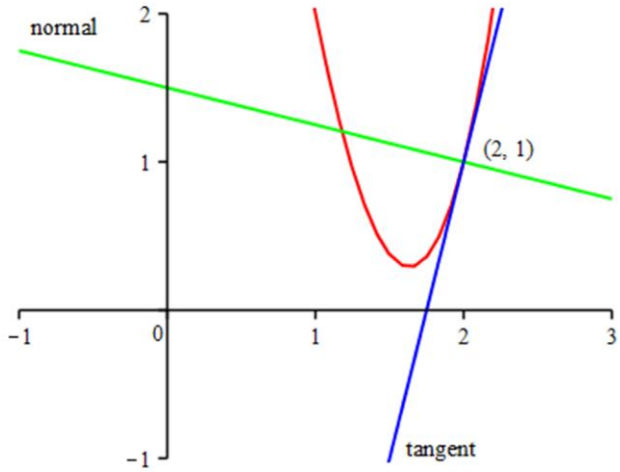
(ii) $\frac{ds}{dt}=\frac{(\sin(t)+1)(4t^3)-t^4\left(\cos(t)\right)}{(\sin(t)+1)^2}$

(iii) $\frac{dw}{dz}=-\frac{1}{4}\left(e^{2z}-(3z+1)^2\right)^{-\frac{5}{4}}\left(2e^{2z}-6(3z+1)\right)$

A Great Video for More Details: https://www.youtube.com/watch?v=lEj3dzj2Doc
A Great Website for More Details:
http://tutorial.math.lamar.edu/Classes/CalcI/ProductQuotientRule.aspx

4) Finding the Equations of the Tangent and Normal to a Curve at a Given Point

Problem: Find the tangent and normal to $y = x^3 - 8x + 9$ at the point where $x = 2$.



Solution: $\dfrac{dy}{dx} = 3x^2 - 8$. At $x = 2$, $y = 1$, $\dfrac{dy}{dx} = 4$.

The tangent line has slope $m = 4$ and passes through $(2,1)$. Using $y - y_0 = m(x - x_0)$, we have $y - 1 = 4(x - 2) = 4x - 8$ and so the tangent has equation $y = 4x - 7$.

The normal is perpendicular to the tangent. If the tangent has slope m, the normal has slope $-\dfrac{1}{m}$, that is, the negative reciprocal of the tangent slope. So, the normal line has slope $m = -\dfrac{1}{4}$. Again using $y - y_0 = m(x - x_0)$, we have

$y - 1 = -\dfrac{1}{4}(x - 2) = -\dfrac{1}{4}x + \dfrac{1}{2}$ and so the normal has equation $y = -\dfrac{1}{4}x + \dfrac{3}{2}$.

Note: Another way of expressing the relationship between the slopes of perpendicular line is that the product of their slopes is -1.

Common error: Using $\dfrac{1}{m}$ for the slope of the normal.

Practice: Find the tangent and normal to $y = \sin(x)$ at the point where $x = \dfrac{\pi}{4}$.

Answer: tangent: $y = \dfrac{1}{\sqrt{2}}x - \dfrac{\pi}{4\sqrt{2}} + \dfrac{1}{\sqrt{2}}$ normal: $y = -\sqrt{2}x + \dfrac{\pi\sqrt{2}}{4} + \dfrac{1}{\sqrt{2}}$

A Great Video for More Details: **https://www.youtube.com/watch?v=Dnw-BjMG7Rg**
A Great Website for More Details:
https://www.mathwarehouse.com/calculus/derivatives/how-to-find-equations-of-tangent-lines.php

5) Exponential Growth and Decay

Problem: A city had a population of 2 million in 2000 and 2.5 million in 2010. The exponential growth model for the population in millions is $P(t) = P_0 e^{kt}$. Assume $t = 0$ corresponds to the year 2000.
(i) Find P_0 and k. (ii) Predict the population in the year 2030
(iii) What is the rate of population growth in 2030?

Solution: (i) Substitute $(0, 2)$ into
$P(t) = P_0 e^{kt}$: $2 = P_0 e^{k(0)} = P_0$ and so $P(t) = 2e^{kt}$.
To find k, use the point $(10, 2.5)$: $2.5 = 2e^{10k}$

$$\therefore\ e^{10k} = \frac{2.5}{2} = 1.25 \Rightarrow 10k = \ln(1.25) \Rightarrow k = \frac{\ln(1.25)}{10} \doteq 0.0223 \text{ and } P(t) = 2e^{\frac{\ln(1.25)}{10}t} \doteq 2e^{0.0223t}$$

(ii) In 2030, $P(30) = 2e^{\frac{\ln(1.25)}{10}(30)} = 2e^{3\ln(1.25)} \doteq 3.9$ million

$$\text{(iii)}\ \ P'(t) = 2\left(\frac{\ln(1.25)}{10}\right)e^{\frac{\ln(1.25)}{10}t} = \frac{\ln(1.25)}{5}e^{\frac{\ln(1.25)}{10}t}$$

The rate of population growth in 2030 $= P'(30) = \dfrac{\ln(1.25)}{5}e^{\frac{\ln(1.25)}{10}(30)} \doteq .0872$ million/year.

Note: $\dfrac{\ln(1.25)}{5}e^{\frac{\ln(1.25)}{10}t} = \dfrac{\ln(1.25)}{5}(e^{\ln(1.25)})^{\frac{t}{10}} = \dfrac{\ln(1.25)}{5}(1.25)^{\frac{t}{10}}$

Common issue: Not knowing how to solve for the exponent in $e^{10k} = \dfrac{2.5}{2} = 1.25$.

Practice: If the rate of decay of a certain substance is directly proportional to the amount remaining, it can be shown that the model for the amount present after t days is $A(t) = A_0 e^{kt}$. The half-life for this substance is 150 days.
(i) If the sample initially weighs 30 grams, after how many days will the sample have degraded 90%?
(ii) What is the rate of decay on day 100?

Answer: (i) $k = \dfrac{\ln(.5)}{150} \doteq -.00462$ so that $A(t) = 30e^{\frac{\ln(.5)}{150}t}$. For 10% remaining, that is, 3 grams, $t \doteq 498$ days.

(ii) $A'(100) = 30\left(\dfrac{\ln(.5)}{150}\right)e^{\frac{\ln(.5)}{150}(100)} \doteq -0.0873$ grams/day. (The answer is negative because the amount is decreasing.)

A Great Video for More Details: https://www.youtube.com/watch?v=j4jSywmXrcU
A Great Website for More Detail:
https://www.mathsisfun.com/algebra/exponential-growth.html

6) Vertical and Horizontal Asymptotes

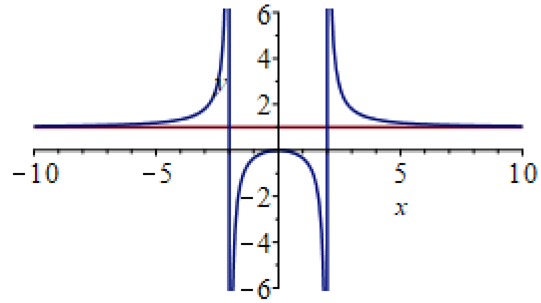
Problem: Find the vertical and horizontal asymptotes of $y = \dfrac{x^2}{x^2-4}$.

Solution: $y = \dfrac{x^2}{x^2-4} = \dfrac{x^2}{(x-2)(x+2)}$

We find vertical asymptotes in a rational function *with no common factors in the top and bottom* by finding values of x where the denominator is 0. Here, the vertical asymptotes are $x=2$ and $x=-2$.

We find horizontal asymptotes by examining the behavior of the function as $x \to \infty$ and $x \to -\infty$. Here,

$$\lim_{x\to\pm\infty} \frac{x^2}{x^2-4} = \lim_{x\to\pm\infty} \frac{x^2}{x^2-4} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\pm\infty} \frac{1}{1-\frac{4}{x^2}} = \frac{1}{1-0} = 1$$



and so there is a horizontal asymptote of $y=1$.

Note: Many non-rational functions have vertical asymptotes. For example, $y=\ln(x)$ has vertical asymptote $x=0$.

Common error: Concluding $y = \dfrac{x-3}{x^2-9}$ has a vertical asymptote at $x=3$.

Practice: Find the vertical and horizontal asymptotes of $y = \dfrac{x+5}{x^3+1}$.

Answer: Vertical asymptote: $x=-1$ Horizontal asymptote: $y=0$

A Great Video for More Details: https://www.youtube.com/watch?v=P0ZgqB44Do4
A Great Website for More Detail:
https://www.mathsisfun.com/algebra/asymptote.html

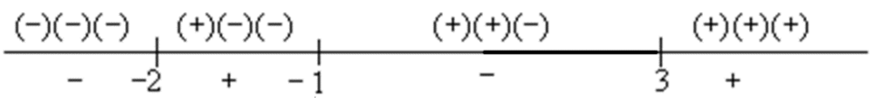
7) Using the Derivative to Determine Where a Function is Increasing and Where it is Decreasing

Problem: If $\dfrac{dy}{dx}=\dfrac{x-3}{(x+2)(x+1)}$, determine the intervals on which y is increasing and those on which it is decreasing.

Solution:

When $\dfrac{dy}{dx}>0$, y increases and when $\dfrac{dy}{dx}<0$, y decreases. The only x values where $\dfrac{dy}{dx}$ can change sign (that is, go from $-$ to $+$ or from $+$ to $-$) are $x=-2,-1,\ 3$.

Using a number line analysis, we find



$\therefore$ y increases on intervals $(-2,-1)$ and $(3,\infty)$ and decreases on intervals $(-\infty,-2)$ and $(-1,3)$.

Note: This method only works when you have completely factored the $\dfrac{dy}{dx}$.

Common error: Not realizing that if a factor $x-a$ is to an **even** exponent in the derivative, then the sign of the derivative does not change at $x=a$.

Practice: If $\dfrac{dy}{dx}=\dfrac{xe^x(x-4)^2}{(x+5)^3}$, determine the intervals on which y is increasing and those on which it is decreasing.

Answer: Increasing: $(-\infty,-5),(0,4),(4,\infty)$ Decreasing: $(-5,0)$. Note at $x=4$, $\dfrac{dy}{dx}=0$.

(Some sources would argue that y increases on the interval $(0,\infty)$, even though $\dfrac{dy}{dx}=0$ at $x=4$. The argument is that the function increases on an interval if whenever in the interval $x_1<x_2$, then $y_1<y_2$. With this definition, you could conclude that the answer is: Increasing on $(-\infty,-5),[0,\infty)$ Decreasing on $(-5,0]$)

A Great Video for More Details: **https://www.youtube.com/watch?v=Dyl7jPlJXOM**
A Great Website for More Detail: https://www.khanacademy.org/math/ap-calculus-ab/ab-diff-analytical-applications-new/ab-5-3/a/increasing-and-decreasing-intervals-review

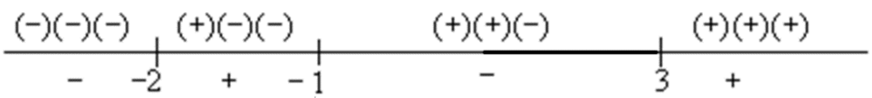
8) Using the Second Derivative to Determine where a Function is Concave Up and where it is Concave Down

Problem: If $\dfrac{d^2y}{dx^2} = \dfrac{x-3}{(x+2)(x+1)}$, determine the intervals on which y is concave up and those on which it is concave down.

Solution:

When $\dfrac{d^2y}{dx^2} > 0$, y is concave up and when $\dfrac{d^2y}{dx^2} < 0$, y is concave down. The only x values where $\dfrac{d^2y}{dx^2}$ can change sign (that is, go from $-$ to $+$ or from $+$ to $-$) are $x = -2, -1, 3$.

Using a number line analysis, we find



$\therefore y$ is concave up on intervals $(-2,-1)$ and $(3,\infty)$ and concave down on intervals $(-\infty,-2)$ and $(-1,3)$.

Note: This method only works when you have completely factored $\dfrac{d^2y}{dx^2}$.

Common error: Not realizing that if a factor $x-a$ in the second derivative is to an **even** exponent, then the sign of the second derivative does not change at $x=a$.

Practice: If $\dfrac{d^2y}{dx^2} = \dfrac{x^{\frac{1}{3}}(x+1)^{\frac{2}{3}}}{(x-5)^3}$, determine the intervals on which y is concave up and those on which it is concave down.

Answer:

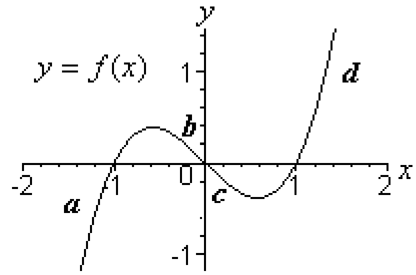
Concave up: $(-\infty,-1),(-1,0),(5,\infty)$ Concave down: $(0,5)$. Note at $x=-1$, $\dfrac{d^2y}{dx^2}=0$.

(Some sources would argue that y is concave up on the interval $(-\infty,0)$, even though $\dfrac{d^2y}{dx^2}=0$ at $x=-1$. The argument is that the function is concave up on an interval if whenever in the interval, $x_1 < x_2$, then $f'(x_1) < f'(x_2)$. With this definition, you could conclude that the answer is:
Concave up on $(-\infty,0],(5,\infty)$ Concave down on $[0,5)$)

A Great Video for More Details:
https://www.youtube.com/watch?v=15awMHeP1Yc
A Great Website for More Detail: https://www.khanacademy.org/math/ap-calculus-ab/ab-diff-analytical-applications-new/ab-5-6b/a/concavity-review

9) Using the First and Second Derivatives to Determine the Shape of a Function

Problem: Each y', y'' box below matches the conditions for one of the four shapes a, b, c, d on the graph. Match them up.



$y' > 0$ $y'' > 0$	$\square$	$y' > 0$ $y'' < 0$	$\square$	$y' < 0$ $y'' > 0$	$\square$	$y' < 0$ $y'' < 0$	$\square$

Solution:

***a* increasing, concave down** $y' > 0 \quad y'' < 0$
***b* decreasing, concave down** $y' < 0 \quad y'' < 0$
***c* decreasing, concave up** $y' < 0 \quad y'' > 0$
***d* increasing, concave up** $y' > 0 \quad y'' > 0$

$y' > 0$ $y'' > 0$	***d***	$y' > 0$ $y'' < 0$	***a***	$y' < 0$ $y'' > 0$	***c***	$y' < 0$ $y'' < 0$	***b***

Note: Another way of expressing the interpretation of
$y' > 0 \quad y'' > 0$: As x goes up, y goes up. As x goes up, y' goes up.

Common error: Confusing the shape determined by
$y' < 0 \quad y'' > 0$ with that determined by $y' < 0 \quad y'' < 0$.

Practice: What does the graph look like if (i) $y' > 0$ and $y'' = 0$ (ii) $y' = 0$ and $y'' = 0$.

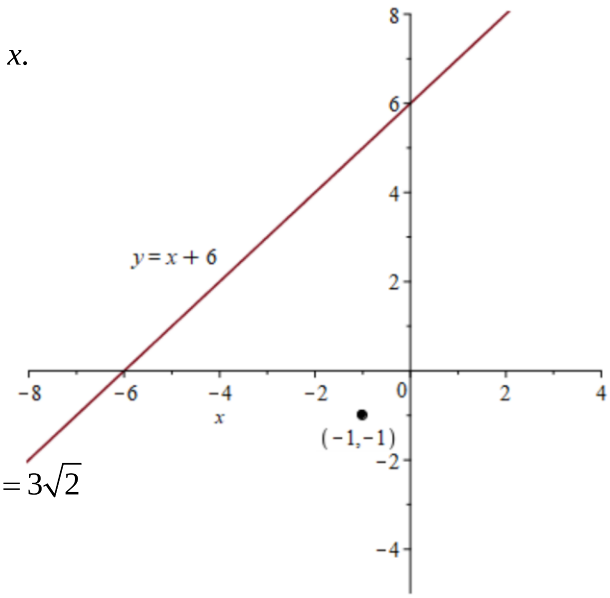
Answer: (i) a straight line with positive slope (ii) a horizontal line

A Great Video for More Details:
https://www.youtube.com/watch?v=s4WCL907jrU&t=347s
A Great Website for More Detail: Couldn't find one. Sorry!

10) Optimization

Problem: Find the minimum distance between the point $(-1,-1)$ and the line $y = x + 6$.

Solution: Let (x, y) be a point on $y = x + 6$. To minimize the distance between (x, y) and $(-1,-1)$, we can minimize the square of the distance, which will make the derivative easier!
Let L be the distance between (x, y) and $(-1,-1)$. Then $L^2 = (x-(-1))^2 + (x+6-(-1))^2 = (x+1)^2 + (x+7)^2$



To minimize L^2, take the derivative with respect to x.

$$\frac{d(L^2)}{dx} = 2(x+1) + 2(x+7) = 4x + 16 = 4(x+4)$$

The derivative is 0 at $x = -4$. Since it is negative for $x < -4$ and positive for $x > -4$, we have an absolute minimum at $x = -4$.

On the line, when $x = -4, y = 2$.
$\therefore$ The minimum distance is

$$L = \sqrt{(-4-(-1))^2 + (2-(-1))^2} = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

Note: $\frac{d^2(L^2)}{dx} = 4 > 0$
and so by the second derivative test, the minimum distance occurs for $x = -4$.

Common issue: Taking the derivative of
$L = \sqrt{(x-(-1))^2 + (x+6-(-1))^2} = \sqrt{(x+1)^2 + (x+7)^2}$
This is not wrong but much messier.

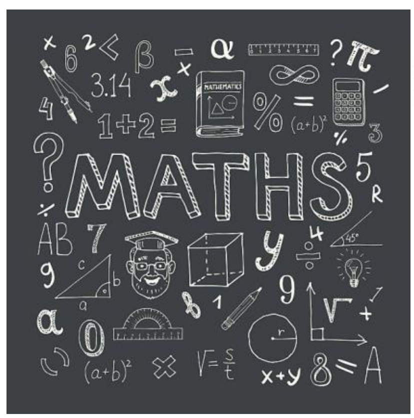
Practice: We have 45 m^2 of material to build a box with a square base and no top. Determine the dimensions of the box that will maximize the enclosed volume.

Answer: The base is $\sqrt{15}$ metres and the height is $\frac{\sqrt{15}}{2}$ metres.

A Great Video for More Details: **https://www.youtube.com/watch?v=DdCfufivnjI**
A Great Website for More Detail:
http://tutorial.math.lamar.edu/Problems/CalcI/Optimization.aspx

Solutions Part 11

Applications and Extras



1) Applications of Sinusoidal Functions

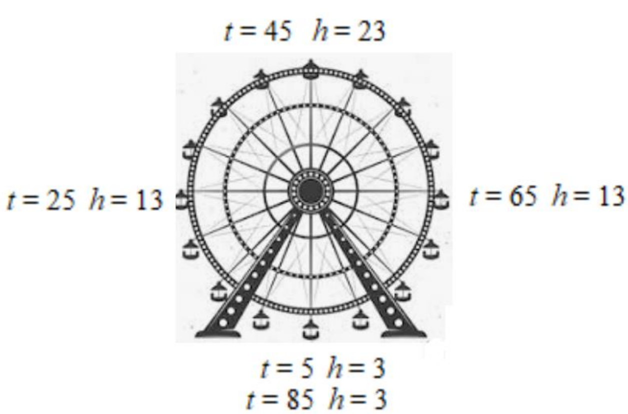
Problem: Max is on a Ferris Wheel that has a diameter of 20 metres. He reaches the lowest part of the ride 5 seconds after getting on, and the highest part at 45 seconds so that a complete rotation takes 80 seconds. The lowest part of the ride is 3 metres above the ground. Determine a cosine function that models his ride on the Ferris Wheel.

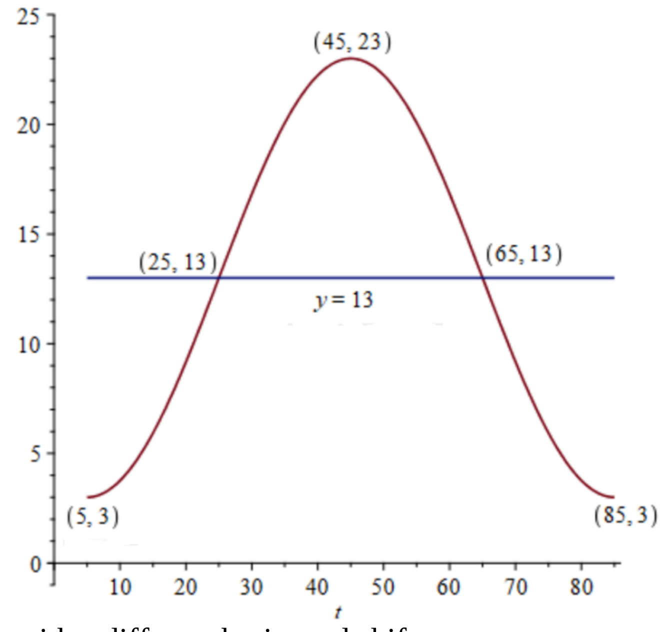
Solution: The motion can be modelled with a cosine function because each rotation of the Ferris Wheel, starting at the bottom, goes from minimum at $t=5$ seconds to maximum $t=45$ back to the minimum at $t=85$ and repeats till the ride ends! So, $h(t)=A\cos(B(t+C))+D$

The amplitude A is the radius of the circle $\Rightarrow |A|=\frac{20}{2}=10$

Let's begin a complete cycle with $t=5.$ Then we are starting at the minimum of the cosine function and so $A=-10.$ The period is $\frac{360}{B}.$ It takes 40 seconds for half of the trip, so 80 seconds for a complete rotation. Therefore, $\frac{360}{B}=80\Rightarrow B=\frac{360}{80}=4.5$

C is the horizontal shift. Here, the bottom of the cosine function (remember, A is negative) is at $t=5$, so we have shifted 5 seconds to the right. Therefore, $C=-5.$
D is the vertical shift and graphically, it is the horizontal axis of symmetry. In this case, it passes through the center of the wheel which is the radius + the height from the ground, that is, $10+3=13.$ Therefore, $h(t)=-10\cos(4.5(t-5))+13.$





Note: We could model this using a sine function with a different horizontal shift: $h(t)=-10\sin(4.5(t+15))+13.$ So, here, for example, $h(5)=-10\sin(4.5(5+15))+13=-10\sin(90)+13=3.$

Common error: Miscalculating the period or *B* value by misunderstanding the information given. Making the function positive when it has been flipped.

Practice: Find the period, amplitude, phase shift, and horizontal axis of symmetry of the following equations:

(i) $y = -2\cos(4(x - 60))$ (ii) $y = \frac{1}{2}\sin(x + 180) + 4$ (iii) $y = \cos(2x + 90) - 7$

Answer:

(i) period $= 90^{\circ}$ amplitude = 2 phase shift $= 60^{\circ}$, right axis of symmetry is $y = 0$

(ii) period = 360° amplitude = $\frac{1}{2}$ phase shift $= 180^{\circ}$, left axis of symmetry is $y = 4$

(iii) period $= 180^{\circ}$ amplitude = 1 phase shift $= 45^{\circ}$, left axis of symmetry is $y = -7$

A Great Video for More Detail: https://www.youtube.com/watch?v=clXSqjs1wgQ
A Great Website for More Detail: https://www.mathsisfun.com/algebra/amplitude-period-frequency-phase-shift.html

2) Solving Word Problems

Problem: A cookie recipe has three times the amount of flour as it does butter and a bowl comfortably fits 8 cups. If the other ingredients take up ¼ of the mix, how much flour and how much butter should you combine to comfortably fit in the bowl?

Solution: Let *f* represent cups of flour and let *b* represent cups of butter.
If flour is 3 times the amount of butter then E1: $f = 3b$,
If the total is 8 cups and ¼ of that is other ingredients, then $8 - \frac{8}{4} = 6$ cups is butter and flour. E2: $b + f = 6$

Substitute E1 into E2 to get $b + 3b = 6 \Leftrightarrow 4b = 6 \Leftrightarrow b = 1.5$ cups
Substitute b =1.5 into E1 (or E2): $f = 3(1.5) = 4.5$ cups
Therefore, you should use 1.5 cups of butter and 4.5 cups of flour.

Note: Word problems can be difficult to understand at times so always read the question more than once. It helps to underline or rewrite important parts in order to pull the math out of the words.

Common error: Forgetting "let" statements. These really clarify what you need to find both for you and for others who needs to follow your reasoning.

Practice: Solve the following equations:
(i) $2y = 3x - 6$ and $13 = 4x - y$ (ii) $3b = \frac{a}{2} + 10$ and $\frac{b}{4} = \frac{a}{2} + \frac{9}{2}$

Answer: (i) $x = 4,\ y = 3$ (ii) $a = -8,\ b = 2$

A Great Video for More Detail: https://www.youtube.com/watch?v=lhQuiC9de98
A Great Website for More Detail: https://www.onlinemathlearning.com/two-variable-word-problems.html

3) Length and Midpoint of the Line Joining Two Points and the Point of Intersection of Two Lines

Problem: (i) Find the length and midpoint of the line that connects the points $(12,-10)$ and $(15,-3)$.
(ii) What is the point of intersection between the lines $y = 2x + 3$ and $3y = 12x + 27$?

Solution: (i) Length = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Midpoint = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Substitute $x_1 = 12, x_2 = 15$ and $y_1 = -10, y_2 = -3$ into the formulas:
Length = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(15 - 12)^2 + (-3 - -10)^2} = \sqrt{(3)^2 + (7)^2}$
$= \sqrt{9 + 49} = \sqrt{58}$

Midpoint = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{15+12}{2}, \frac{-10 - 3}{2}\right) = \left(\frac{27}{2}, -\frac{13}{2}\right)$

(ii) The point of intersection is a point that satisfies both equations. We just solve for the unknown variables. Using elimination, multiply $y = 2x + 3$ by 3 so that it has the same coefficient for y as $3y = 12x + 27$.

$y = 2x + 3$ multiplied by 3 becomes $3y = 6x + 9$. Subtract to eliminate y.
$3y = 12x + 27$
$-\ (3y = 6x + 9)$
=> $0 = 6x + 18$ Solve for x: $0 = 6x + 18 \Leftrightarrow 6x = -18 \Leftrightarrow x = -3$
Now substitute $x = -3$ into either of the original equations.
$y = 2(-3) + 3$ and so $y = -3$. The point of intersection is $(-3,-3)$.

Note: For the length of a line, you are essentially calculating the hypotenuse of a right triangle. The intersection of two lines can be determined using the substitution method as well as the elimination method.

Common error: Not combining negative numbers properly. Doing $x_1 + y_1$ instead of $x_1 + x_2$ when calculating the midpoint. Choosing the wrong operation (+/-) for elimination.

Practice:
(i) What is the length and midpoint of the line that connects the points (2,4) and (3,6)?
(ii) What is the point of intersection between $2y = 4x - 2$ and $4y = 8x + 7$?

Answer: (i) Length = $\sqrt{5}$ Midpoint = $\left(\frac{5}{2}, 5\right)$ (ii) These lines are parallel and not "coincident" (that is, they are not the same line!) and so they do NOT intersect!

A Great Video for More Detail: https://www.youtube.com/watch?v=Ez_-RwV9WVo
A Great Website for More Detail: https://www.freemathhelp.com/length-line-segment.html

4) Optimization

Problem: Hannah is constructing a rectangular garden with a fence around it. She has 32 metres of fencing to put along the edges and she will need to leave a 2 metre gap for the gate. What dimensions should be used to maximize the area of the garden?

Solution: Develop equations for the perimeter and area. Let l be length of the garden and w be width in metres. The perimeter would include the 2 metre gap with the fencing.
$P = 2w + 2l = 32 + 2 = 34$ m and $Area = lw$

Solve for l in the perimeter equation: $l = \frac{(34 - 2w)}{2} = 17 - w$

Substitute into the area equation and simplify => Area $= w(17 - w) = -w^2 + 17w$
Since the w^2 coefficient is negative, we know that the optimal value is a maximum. Set derivative equal to 0.

$$\frac{dA}{dw} = -2w + 17 = 0 => 17 = 2w => w = \frac{17}{2} \text{ m}$$

Substitute $w = \frac{17}{2}$ into $2w + 2l = 34 => 2\left(\frac{17}{2}\right) + 2l = 34 => 17 + 2l = 34$

$=> 2l = 17 => l = \frac{17}{2}$ m

Therefore, the dimensions of the garden should be 8.5 metres by 8.5 metres in order to maximize the area of the garden.

Note: The maximum area of a rectangle will always be given by a square. The sign in front of the quadratic function will tell you whether the optimal value is a maximum (-ve) or minimum (+ve). We could have solved this problem by completing the square.

Common error: Trying to solve this problem through substitution (like with linear equations).

Practice: Determine the optimal point for each of the following and state whether it is a maximum or minimum.
(i) $y = 3x^2 - 12x + 4$ (ii) $6s = -18t^2$ (iii) $4 + w - 20v = 5v^2$

Answer: (i) $(2, -8)$ minimum (ii) $(0, 0)$ maximum (iii) $(-2, -24)$ minimum

A Great Video for More Detail: https://www.youtube.com/watch?v=DdCfufivnjI
A Great Website for More Detail:
http://www.datagenetics.com/blog/august12014/index.html

5) Mapping Points in a Transformation

Problem: For each transformation, identify the parent function. Then in (i), find the images of points $(-2,4)$ and $(4,16)$ on the graph of *e* under the transformation *f*. In (ii), find the images of points (10,0.1), (0.5, 2) on the graph of *g* under the transformation *h*. Use mapping notation.

(i) $e(x) \to f(x) = -\left(\frac{x}{3}+6\right)^2 + 4$ (ii) $g(x) \to h(x) = \frac{2}{x+1}$

Solution: (i) The parent function is $e(x) = x^2$. We want the coordinates of $(-2,4)$, $(4,16)$ after applying the transformation *f*. Factor out the *x* coefficient to get

$f(x) = -\left(\frac{1}{3}(x+18)\right)^2 + 4$

The operations on *x*: multiply by 3 and then subtract 18. (In effect, we are solving $\frac{1}{3}(x+18) = -2.$ to find where *x* came from in the domain of *f*.)

The operations on *y*: multiply by -1 and add 4. We are building up *y* as specified by *f*.

$(-2,4) \to (-2(3) - 18\,, 4(-1) + 4) = (-6 - 18\,, -4 + 4) = (-24,0)$

$(4,16) \to (4(3) - 18\,, 16(-1) + 4) = (12 - 18\,, -16 + 4) = (-6,-12)$

Final mapping notations: $(-2,4) \to (-24,0)$ $(4,16) \to (-6,-12)$

(ii) The parent function is $g(x) = \frac{1}{x}$. We want the coordinates of $(10, 0.1)$ and $(0.5, 2)$ after applying *h*. The operation on *x*: subtract 1. The operation on *y*: multiply by 2.

$(10, 0.1) \to (10 - 1, 0.1(2)) = (9, 0.2)$ $(0.5, 2) \to (0.5 - 1,\ 2(2)) = (-0.5,4)$

Final mapping notations: $(10,0.1) \to (9, 0.2)$ and $(0.5, 2) \to (-0.5,4)$

Note: Operations in the brackets are performed on the *x* values (horizontal) and anything outside of the bracket affects the *y* values (vertical).

Common error: Forgetting to reverse operations with the *x* values. If done correctly, we reverse adding by subtracting and we reverse multiplying by dividing.

Practice: For each transformation, identify the parent function. Then determine new coordinate pairs from the coordinate pairs given using mapping notations.

(i) $r(x) \to s(x) = \sqrt{x+2} - 7$; $(0,0)$, $(1,1)$, $(4,2)$

(ii) $t(x) \to u(x) = -8|x+2|$; $(-5,5)$, $(0,0)$, $(5,5)$

Answer:

(i) $r(x) = \sqrt{x}$ $(0,0) \to (-2,-7)$, $(1,1) \to (-1,-6)$, and $(4,2) \to (2,-5)$

(ii) $t(x) = |x|$ $(-5,5) \to (-7,-40)$, $(0,0) \to (-2,0)$, and $(5,5) \to (3,-40)$

A Great Video for More Detail: https://www.youtube.com/watch?v=gYYrqjMbAU0
Another Great Video: https://www.youtube.com/watch?v=5qI0QcQsjXQ

6) Inverse functions

Problem: Find $f^{-1}(x)$ for each of the following and state the new Domain and Range.

(i) $f(x) = \sqrt[3]{x} + 19$ (ii) $f(x) = \dfrac{1}{x+3}$

Solution: (i) Interchange x and y to get $x = \sqrt[3]{y} + 19$ and solve for y to get $f^{-1}(x)$.

$x = \sqrt[3]{y} + 19 \Rightarrow x - 19 = \sqrt[3]{y} \Rightarrow (x-19)^3 = y$

Therefore: $f^{-1}(x) = (x-19)^3$ $D = \{x \in \mathbb{R}\}$ and $R = \{y \in \mathbb{R}\}$

(ii) Interchange x and y to get $x = \dfrac{1}{y+3}$ and solve for y to get $f^{-1}(x)$.

$x = \dfrac{1}{y+3} \Rightarrow y+3 = \dfrac{1}{x} \Rightarrow y = \dfrac{1}{x} - 3$ Therefore, $f^{-1}(x) = \dfrac{1}{x} - 3$.

Domain $= \{x \in \mathbb{R} | x \neq 0\}$ and Range $= \{y \in \mathbb{R} | y \neq -3\}$

Note: Picture the inverse by visualizing the original function *reflected* in the line $y = x$. You could also choose points on the original and switch x and y to get points on the inverse.

Common error: Making mistakes with the algebra/rearranging after switching x and y.

Practice: Find $f^{-1}(x)$ for each of the following

(i) $f(x) = 5x + 7$ (ii) $f(x) = \dfrac{3}{x}$ (iii) $f(x) = \sqrt[4]{x-3} + 2$

Answer: (i) $f^{-1}(x) = \dfrac{x-7}{5}$ (ii) $f^{-1}(x) = \dfrac{3}{x}$ (iii) $f^{-1}(x) = (x-2)^4 + 3$

A Great Video for More Detail: https://www.youtube.com/watch?v=-l7U3dEAGa4
A Great Website for More Detail:
https://www.purplemath.com/modules/invrsfcn3.htm

7) Applications of Quadratic Functions

Problem: A ball is thrown along a path represented by $h(t) = -5t^2 + 40t + 2$ where $h(t)$ is the height in metres above the ground and t is time in seconds, $t \geq 0$.
(i) What is the maximum height that the ball reaches? (ii) At what time does the ball hit the ground? (iii) What is the initial height of the ball?

Solution: (i) The maximum height would be at the vertex of the parabola. Change the equation from standard form, to vertex form by completing the square.

$$h(t) = -5t^2 + 40t + 2 = -5(t^2 - 8t) + 2 = -5(t^2 - 8t + 16 - 16) + 2$$
$$= -5(t^2 - 8t + 16) + 80 + 2 = -5(t-4)^2 + 82.$$

The vertex is (4,82).
Therefore the maximum height it 82 metres, occurring at $t = 4$ seconds.
(ii) The ball hits the ground when $h(t) = -5t^2 + 40t + 2 = 0$
Use the quadratic formula to solve for t. $a = -5, b = 40, c = 2$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-40 \pm \sqrt{(-40)^2 - 4(-5)(2)}}{2(-5)} = \frac{-40 \pm \sqrt{1600 + 40}}{-10} = \frac{-40 \pm \sqrt{1640}}{-10}$$

Since $t \geq 0$, $t = \dfrac{-40 - \sqrt{1640}}{-10} \doteq 8.05$ seconds.
(iii) When $t = 0$, $h(0) = -5(0)^2 + 40(0) + 2 = 2$
Therefore the ball was initially 2 metres above the ground.

Note: Write $h(t) = A(t - B)^2 + C$. If $A > 0$, then C is a minimum, occuring at $t = B$.
If $A < 0$, C is a maximum.

Common error: When converting to vertex form, forgetting to multiply the term in the bracket with the coefficient before combining it with the last term.

Practice: If a rocket travels along a path represented by $h(t) = -4.9t^2 + 49t$ where $h(t)$ is height from the ground in metres and t is time in seconds.
(i) When does it reach the maximum height? (ii) What is the launch height (ie., when $t = 0$)? (iii) At what time t does it hit the ground?

Answer: (i) 5 seconds (ii) 0 metres (iii) 10 seconds

A Great Video for More Detail: https://www.youtube.com/watch?v=gJudLQquYGo
A Great Website for More Detail: https://www.onlinemathlearning.com/quadratic-equations-word-problems.html

8) Simple and Compound Interest

Problem: (i) You invest \$1000 into an account that earns 6% interest compounded monthly. Determine the interest earned after 10 years.
(ii) How much interest would you earn if you were only receiving simple interest each month? (Simple Interest: $I = Prt$)

Solution: (i) $A = P\left(1+\frac{r}{n}\right)^{nt}$, where A is the final amount, P is the initial investment, r is the interest rate per year, n is the number of times per year that the interest is compounded, and t is time in years. Here, $P = 1000$, $r = .06$, $n = 12$, and $t = 10$.

Therefore, $A = 1000\left(1+\frac{.06}{12}\right)^{12\times 10} = 1000\left(1+\frac{.06}{12}\right)^{120} = 1000(1.005)^{120} \doteq 1819.40$

Therefore, the interest earned $= 1819.40 - 1000 = \$819.40$

(ii) $I = Prt = 1000\left(\frac{.06}{12}\right)(120) = \$600$

Note: Convert r to a decimal by dividing by 100.

Common error: Forgetting to convert the percentage to a decimal. Using the numbers of years t without considering the number of times the interest is compounded per year.

Practice: (i) How much **simple** interest will you earn if you put \$2500 into an account for 20 years that earns 3.5% each year?
(ii) If you put \$8000 into an account that earns 3% interest compounded every 3 months, how much will you have after 30 years?

Answer: (i) \$1750.00 (ii) \$19610.86

A Great Video for More Detail: https://www.youtube.com/watch?v=OQ9Mv2jwQWo
A Great Website for More Detail: https://ca.ixl.com/math/grade-12/compound-interest-word-problems

9) Mean, Median, and Mode

Problem: Find the mean, median, and mode of the following sets of data:
(i) 5.5, 2.1,7.4, 10.0, 7.3, 2.1, 9.8 (ii) 2, -3, 4, -6, 9, 1, -1, 10, -10, 12

Solution: (i) mode = 2.1 because it is repeated the most.
The median is the middle number when the set is put in ascending order:
2.1, 2.1, 5.5, 7.3, 7.4, 9.8, 10.0 => median = 7.3.
The mean is the sum of the numbers divided by the number of numbers in the set:

$$\text{mean} = \sum_{n=1}^{7} x_n = \frac{2.1+2.1+5.5+7.3+7.4+9.8+10}{7} \doteq 6.3$$

(ii) There is no mode when no numbers are repeated.
For the median, put the numbers in ascending order: $-10, -6, -3, -1, 1$, 2**,** 4, 9, 10, 12.
When there an even number of numbers, find the average of the middle two:

$$\text{median} = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$$\text{mean} = \sum_{n=1}^{10} x_n = \frac{-10-6-3-1+1+2+4+9+10+12}{10} = 1.8$$

Note: Be sure to order the terms in descending or ascending order before finding the median. *n* in the mean equation is the number of terms in the set.

Common error: Forgetting to put the terms in order when evaluating the median. Forgetting the negatives when finding the sum for the mean.

Practice: Find the mean, median, and mode of the following sets of data:
(i) 1, 1, 1, 1, 3, 4, 6, 6, 9, 14 (ii) -4, 5, 9, -10, -4, 2, -8, 5, -4
(iii) 0.2, 0.7, 0.2, 1.5, 3.7, 0.2

Answer: (i) Mean = 4.6 Median $= 3.5$ Mode $= 1$
(ii) Mean $= -1$ Median $= -4$ Mode $= -4$
(iii) Mean = 1.08 Median $= .45$ Mode $= .2$

A Great Video for More Detail: https://www.youtube.com/watch?v=36FyCxJKwsU
A Great Website for More Detail:
https://www.thoughtco.com/the- mean-median-and-mode-2312604